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AN EARTH-BASED COORDINATE CLOCK NETWORK

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AN EARTH-BASED COORDINATE CLOCK NETWORK

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This paper investigates some of the possible operational procedures for synchronizing clocks at fixed sites spread around on the earth's surface, to within a 1 nanosecond level of accuracy. Since a common synchronization procedure is by transport of standard clocks in commercial jet airline flights, and most of the effects we shall discuss are fractional corrections to the elapsed time, as a criterion of whether an effect is significant at the 1 nanosecond level we take for comparison purposes an elapsed time T_c = 10 hours. This is a typical time for an intercontinental airplane flight. Analysis of a number of effects which might affect clock synchronization is carried out within the framework of general relativity. These effects include the gravitational fields of the earth, sun, and moon, and orbital motion, rotation, and flattening of the earth. It is shown that the only significant effects are due to the gravitational field and rotation of the earth, and motion of the transported clocks. Operational procedures for construction of a synchronized coordinate clock network using light signals and transported standard clocks are discussed and compared.

Key words: Clocks; coordinate time; general relativity; time

Introduction

The purpose of this paper is to investigate some of the possible operational procedures for synchronizing clocks at various fixed sites spread around on the earth's surface. There are a number of effects which could conceivably play a role in the synchronization of clocks--among these are the gravitational fields of the earth, moon, and sun, and the orbital motions, rotation, and flattening of the earth. Since most of the effects we shall discuss turn out to be fractional corrections to elapsed time (that is, the corrections to the time are proportional to the time), as a criterion of whether an effect is significant at the level of 1 nanosecond (1 ns), we shall take for comparison purposes an elapsed time T_c = 10 hours, which corresponds to a typical time for an intercontinental airplane flight. This "comparison flight" will, where appropriate, be assumed to occur at an altitude of h_c = 12,000 meters and at a speed of 450 meter/sec. The reason for selection of these parameters is that a fairly common synchronization procedure will be by transport of standard clocks in commercial jet airline flights [1].

The analysis will be carried out within the framework of the general theory of relativity; we shall not discuss any modification of procedure which might be necessary in some other gravitational theory such as that due to Brans and Dicke [2].

The basic assumptions and results of the general theory of relativity, which we shall repeatedly use, are as follows (we use the notation and sign conventions of Weber [3]):

1) There exists a metric tensor $g_{\mu\nu}$ (μ,ν = 1, 2, 3) such that the space-time interval ds defined by

$$-ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{1}$$

is invariant with respect to arbitrary coordinate transformations.

2) Propagation of light rays along the path element dx^{μ} is described by the vanishing of ds:

$$0 = g_{\mu\nu}dx^{\mu}dx^{\nu} . \tag{2}$$

3) The proper time elapsed on a standard clock transported along the space-time path element dx^{μ} is

$$\frac{1}{c}\, ds = \frac{1}{c}\sqrt{- g_{\mu\nu}dx^{\mu}dx^{\nu}}. \tag{3}$$

4) Of particular significance to this investigation is the interpretation of the coordinate x^0 , which is a global coordinate time having the units of length. It is assumed that at each point of space it is possible to equip an observer with a clock (a coordinate clock) which can be used to measure the coordinate time of any event occurring at that point. The importance of coordinate time may be appreciated by considering, for example, the phenomenon of the redshift of a clock in a static gravitational field. Standard clocks (e.g., atomic clocks) will have different rates depending on the gravitational potential at the position of the clock. In contrast, a coordinate clock will always have the same rate independent of position in the gravitational field.

5) The metric coefficients $g_{\mu\nu}$ may be calculated from the field equations of general relativity. Furthermore, because in the vicinity of the earth's surface all gravitational fields are weak, it will be sufficient to work in the linearized approximation to general relativity in which

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4}$$

with $\eta_{\mu\nu}$ = (-1, 1, 1, 1) the metric tensor of flat space and

$$h_{\mu\nu} << 1$$

so that squares and products of $h_{\mu\nu}$ are negligible.

2. Summary of Results

We have investigated two principal schemes for obtaining a worldwide network of synchronized coordinate clocks. These are: A) a coordinate clock network measuring the global coordinate time x^0, and B) a coordinate clock network with a discontinuity (the "Hafele-Keating" discontinuity) along some line between the poles.

A) A coordinate clook system measuring global coordinate time x^0

In this scheme, it is assumed that the metric tensor is known theoretically in terms of the earth's mass and rotation. One may then, with some selected point on the geoid (the geoid is the equipotential surface at mean sea level) as reference, transport standard clocks from point to point on the earth's surface. The elapsed global coordinate time Δx^0 during the process may be calculated from the equation

$$\Delta x^0 = \int_{path} ds\left[1 - \frac{gh}{c^2} + \frac{v^2}{2c^2} + \frac{\omega a_1 v_E \cos\phi}{c^2}\right] \qquad (5)$$

where ds is the elapsed proper time as measured on the transported clock; g is the acceleration of gravity; v is the ground velocity of the clock having an eastward component v_E; h is the altitude above the geoid; ω is the rotational angular velocity of the earth; a_1 is the earth's equatorial radius, and ϕ is the geographical latitude. The standard clock readings must thus be corrected for redshift, ground speed, and earth rotation before using them to measure coordinate time.

Note that for a clock at rest on the earth, Equation (5) reduces to

$$\Delta x^0 = \Delta s\left[1 - \frac{gh}{c^2}\right] \qquad (6)$$

so that a standard clock at rest on the earth's surface must have its rate corrected for redshift before being used to measure elapsed coordinate time x^0 at that point.

Light signals may be used to perform an equivalent coordinate clock synchronization by using the following formula for the coordinate time elapsed during propagation which involves a correction due to rotation of the earth:

$$\Delta x^0 = \int_{path} d\sigma \left[1 + \frac{\omega a_1 c_E \cos\phi}{c^2}\right]. \qquad (7)$$

In the above expression, $d\sigma$ is the element of proper distance along the path of the light ray, assumed to lie within about 12,000 meters of the geoid, and c_E is the eastward component of the light velocity. The coordinate clock network obtained by this scheme has the following features and disadvantages:

1) Use of transported clocks and light signals as corrected by Equations (5) and (7) will give a mutually consistent network of coordinate clocks.

2) The synchronization procedure gives path-independent results with respect to either transport of clocks or use of light signals.

3) The procedure has the property of transitivity: if clocks A and B are synchronized, then A and C will be synchronized.

4) The coordinate clocks directly measure the global coordinate time x^0.

5) There is no discontinuity in synchronization.

6) A disadvantage is that corrections must be applied to standard clock readings due to redshift (whether the clock is at rest or is transported), ground velocity, and earth rotation.

Equation (5) may also be applied to standard clocks while in motion--for example, while being transported in aircraft.

B) A coordinate clock system with a discontinuity

This scheme has been mentioned by implication in a paper by Schlegel [4]. Here one transports clocks or uses light signals without applying corrections for the earth's rotation. Thus Equation (5) for standard clock transport becomes

$$\Delta x_D^0 = \int_{path} ds \left[1 - \frac{gh}{c^2} + \frac{v^2}{2c^2}\right] . \qquad (8)$$

And for light propagation, Equation (7) becomes

$$\Delta x_D^0 = \int_{path} d\sigma . \qquad (9)$$

This scheme has the following features and advantages:

1) There must be, due to the earth's rotation, a discontinuity in synchronization along some line between the poles given by [5]

$$2\pi\omega a_1^2 \cos^2 \phi/c^2 = 207.4 \cos^2 \phi \text{ ns}. \qquad (10)$$

The existence of this discontinuity somewhat cancels the advantage of not having to correct in Equations (8) and (9) for the earth's rotation.

2) The use of clocks or light signals will give a mutually consistent network of synchronized coordinate clocks as long as the discontinuity is not crossed and transport is along a parallel of latitude.

3) The synchronization procedure is in general path-dependent.

4) The procedure is in general non-transitive.

5) The coordinate clocks do not measure the global coordinate time x^0 of relativity theory.

6) Calculated corrections must be applied, for ground velocity and redshift, to the readings of transported clocks.

C) Investigation of other effects

Quite a number of effects which influence these synchronization procedures were investigated and found to be insignificant at the 1 ns level. These effects will be discussed in detail later, but it is of interest to mention some of them here. They include variations of the gravitational field strength with latitude and altitude, gravitational anomalies, flattening of the earth due to rotation, mass of the atmosphere, gravitational fields due to the sun and moon and orbital motion of the earth. These latter effects are individually of order 10 ns during a typical flight but cancel very precisely.

3. Synchronization in a Static Gravitational Field

To provide a foundation for later discussion, in this section we shall describe the operational procedure by which one could set up a series of coordinate clocks, measuring the global coordinate time x^0, at rest in space. This procedure is worth some discussion because it is more usual in theoretical work to assume the coordinate clocks exist and then to imagine using them to observe and to compare what happens to standard clocks and other devices.

The real situation is different. We actually have available not coordinate clocks but standard clocks whose rate depends on both position and velocity. Furthermore, we know to high precision the positions and motion of the earth, sun, and moon which are the only significant sources of gravitational fields in the neighborhood of the earth's surface. We may, therefore, regard $g_{\mu\nu}$ as known theoretically and the elapsed time on a moving standard clock as experimentally observable. Equation (3) then allows us to compute the elapsed coordinate time at the position of a transported standard clock and hence to set up a coordinate clock at that position.

We now consider this process in detail. We stress the assumption that there exists a global coordinate time x^0. We shall also assume the metric tensor is static,

$$\frac{\partial g_{\mu\nu}}{\partial x^0} = 0, \tag{11}$$

and time orthogonal,

$$g_{0k} = 0, \qquad (k = 1,2,3). \tag{12}$$

We work entirely within the framework of general relativity, in which the invariant ds^2 is now given by

$$-ds^2 = g_{00}(dx^0)^2 + g_{k\ell}dx^k dx^\ell. \tag{13}$$

where repeated latin indices are summed from 1 to 3. The path of a light ray is characterized by

$$ds^2 = 0 \tag{14}$$

and the distance between two events as measured with standard measuring rods is given by

$$d\sigma = (g_{k\ell}dx^k dx^\ell)^{1/2}. \tag{15}$$

From an operational viewpoint, it is possible at any point of space to place a standard clock at rest. One way of achieving this (at least in principle) has been described as follows [6]: Observers using radar arrange to move along the coordinate world lines x^k = const., (k = 1, 2, 3). They do this by adjusting their velocities until each finds that the radar echoes from his neighbors require the same round-trip time at each repetition. Equivalently, each returning echo must show zero Doppler shift.

Let Δs be the invariant proper time between beats of a standard clock placed at rest. This will be the same number no matter where the clock is placed. Then since the clock is at rest, the coordinate time Δx^0 between beats is given by

$$\Delta x^0 = \Delta s \Big/ \sqrt{-g_{00}} \; . \tag{16}$$

This result can be used to determine the rate of a coordinate clock, which measures the coordinate time x^0 , placed at the position of the standard clock. In this expression we may imagine that g_{00} is a function of position determined from general relativity in terms of the distribution of mass. Thus since standard clocks and numerical values of g_{00} are available, Equation (16) provides an operational procedure for setting the rates of coordinate clocks.

The next problem is to initialize the coordinate clocks at different points in space so that the coordinate clock readings provide us with the desired global coordinate times. We shall consider two alternate (but equivalent) procedures for accomplishing this: A) use of light signals, and B) slow transport of standard clocks.

A) Synchronization (Initialization) using light signals

In Figure 1, the lines A and B represent the world lines of two coordinate clocks at rest at positions x^k , $x^k + dx^k$ which we wish to synchronize.

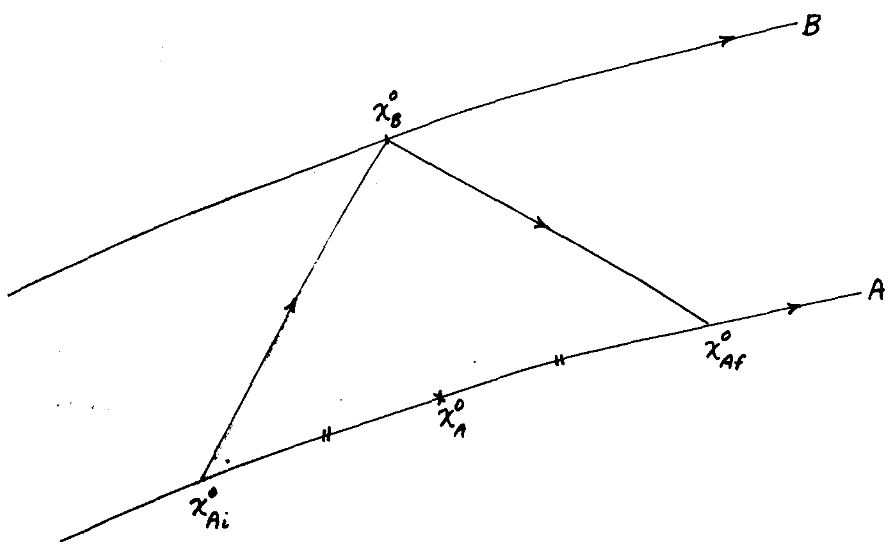

Figure 1

A light signal leaves A at coordinate time $x^0_{A_i}$, coincides with B at x^0_B, and returns to A at $x^0_{A_f}$. The coordinate times $x^0_{A_i}$ and $x^0_{A_f}$ are measured and the coordinate clock at A halfway between the events $x^0_{A_i}$ and $x^0_{A_f}$ reads

$$x^0_A = 1/2\left(x^0_{A_i} + x^0_{A_f}\right) \tag{17}$$

The event at x^0_A should then have the same coordinate time as that at x^0_B, so the coordinate clock at B (or at A) is zeroed so that

$$x^0_B = x^0_A = 1/2\left(x^0_{A_i} + x^0_{A_f}\right). \tag{18}$$

To see the basis for this procedure starting from the expression for the metric, we have for the coordinate time for light to go from A to B,

$$g_{00}\left(dx^0_{go}\right)^2 + d\sigma^2 = 0 \tag{19}$$

and to come back from B to A,

$$g_{00}\left(dx^0_{come}\right)^2 + d\sigma^2 = 0 \tag{20}$$

where $d\sigma$ is the standard distance from A to B. Thus,

$$dx^0_{go} = dx^0_{come} = d\sigma \Big/ \sqrt{-g_{00}} \tag{21}$$

and

$$x^0_B = x^0_{A_i} + d\sigma \Big/ \sqrt{-g_{00}}$$

$$x^0_{A_f} = x^0_B + d\sigma \Big/ \sqrt{-g_{00}}$$

because the metric is static. Thus,

$$x^0_A = 1/2\left(x^0_{A_f} + x^0_{A_i}\right) = 1/2\left(x^0_B + d\sigma \Big/ \sqrt{-g_{00}} + x^0_B - d\sigma \Big/ \sqrt{-g_{00}}\right) = x^0_B, \tag{22}$$

and using Equation (18) gives the desired synchronization procedure.

Equivalently, the above procedure can be used to synchronize a series of coordinate clocks along a path taken by a light ray by writing

$$dx^0 = d\sigma \Big/ \sqrt{-g_{00}} \tag{23}$$

and integrating along the path

$$\Delta x^0 = \int_{\text{path}} d\sigma / \sqrt{-g_{00}} \ . \tag{24}$$

This procedure gives a resulting synchronization which is independent of the path along which the light ray travels; otherwise it would contradict the hypothesis that there exists a global coordinate time. It is also independent of time, since the positions of the coordinate clocks and the numerical values of $g_{\mu\nu}$, are time-independent.

B) <u>Slow transport of standard clocks</u>

We next establish the fact that if a standard clock is moved sufficiently slowly over a small distance past a stationary standard clock and then brought slowly back into coincidence with the stationary clock, the two clocks will undergo the same elapsed proper time. To prove this, we observe that the stationary clock will undergo an elapsed proper time

$$\Delta s = \sqrt{-g_{00}} \, \Delta x^0 \ . \tag{25}$$

The moving clock will undergo the elapsed proper time

$$\Delta s = \int_{\text{path}} \sqrt{-g_{00}} \, dx^0 \left[1 + \ d\sigma^2 \ g_{00} (dx^0)^2 \right]^{1/2}$$

$$\cong \sqrt{-g_{00}} \, \Delta x^0 - 1/2 \int \frac{dx^0}{\sqrt{-g_{00}}} \frac{v^2}{c^2} \ . \tag{26}$$

If the velocity is sufficiently small the second term is negligible. If v is replaced by its maximum value, v_{max}, the condition that the second term in Equation (26) be negligible in comparison with the contribution from h_{00}, the deviation of g_{00} from -1, is

$$\frac{v^2_{max}}{c^2} << \left| 1 + g_{00} \right| \tag{27}$$

Then in the first term of Equation (26) for infinitesimally short paths, g_{00} may be evaluated at any position along the path. This argument establishes the statement in the first paragraph of this section.

Next consider the process of synchronization by means of slow transport of standard clocks; in Figure 2, the solid lines with arrows represent light signals used to synchronize coordinate clocks at A and B repeatedly. We choose

$$x_A^{0''} - x_A^{0'} = x_A^{0'} - x_A^0 \tag{28}$$

and then set the coordinate clocks at point B, an infinitesimal distance away from A, so that

$$x_B^0 = x_A^0 , \qquad x_B^{0'} = x_A^{0'} , \qquad x_B^{0''} = x_A^{0''} \tag{29}$$

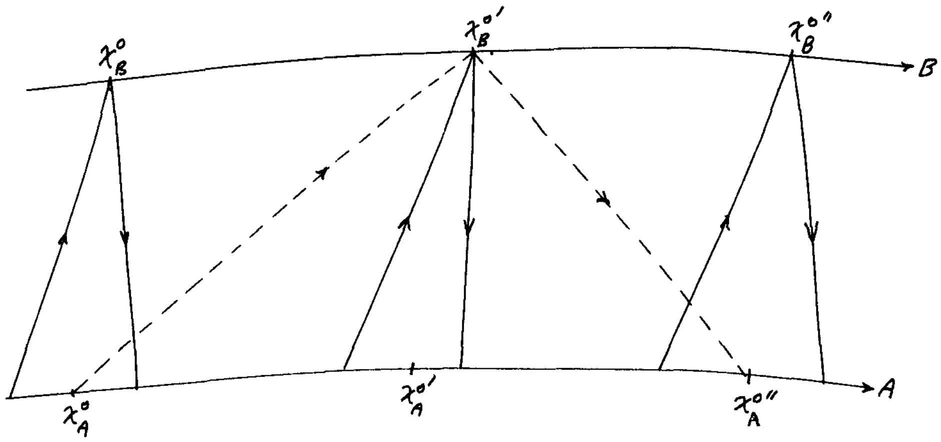

Figure 2

Now starting at coordinate time x_A^0 , cause a standard clock to be transported slowly and uniformly from A to B and back again (dotted line in Figure 2), arriving at B at $x_B^{0'}$ (measured by the coordinate clock at B), and at A at $x_A^{0''}$. The coordinate time required to go from A to B is

$$dx^0 = x_B^{0'} - x_A^0 = x_A^{0'} - x_A^0 = 1/2(x_A^{0''} - x_A^0) \ . \tag{30}$$

Note that $x_B^{0'} - x_A^0$ is a combination of measurements made by coordinate clocks at different locations. In terms of proper time elapsed on a standard clock at A ,

$$dx^0 = 1/2 \frac{s''_A - s_A}{\sqrt{-g_{00}(A)}} \ . \tag{31}$$

If the elapsed proper time on the moving standard clock is Δs , then we have seen already that $\Delta s = s''_A - s_A$. Thus,

$$dx^0 = 1/2 \ \Delta s \Big/ \sqrt{-g_{00}(A)} \ . \tag{32}$$

By symmetry, $1/2 \ \Delta s$ is the proper time elapsed on the slowly moving standard clock in going from A to B , which we shall denote by ds . Thus,

$$dx^0 = ds \Big/ \sqrt{-g_{00}} \tag{33}$$

where to first order in infinitesimal quantities, g_{00} can be evaluated at any point from A to B .

To extend this procedure to clocks which are arbitrarily distant from each other, imagine carrying a standard clock very slowly from one point to another along a path for which g_{00} is known. Then

$$\Delta x^0 = \int_{\text{path}} \frac{ds}{\sqrt{g_{00}}} . \tag{34}$$

The following argument establishes the equivalence of the two methods of coordinate clock synchronization discussed above. In Figure 3, let coordinate clocks at A and B , a finite distance apart, be synchronized by means of light signals and let the dotted line represent the path of a standard clock carried from A to B . Synchronization signals are

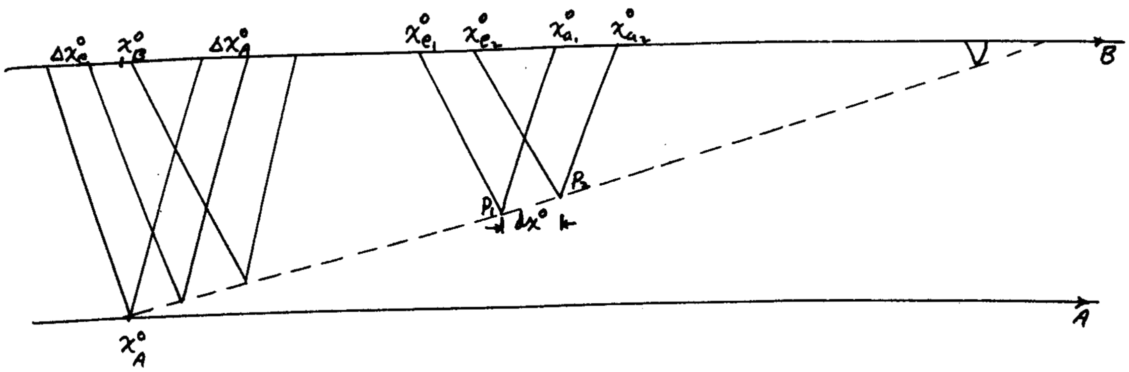

Figure 3

sent out from B at intervals Δx_e^0 and arrive back after reflection from the slowly moving clock at intervals Δx_a^0 . At some point along the path, bounded by reflections of signals which were emitted from B at coordinate times $x_{e_1}^0$ and $x_{e_2}^0$ and which arrive back at B at coordinate times $x_{a_1}^0$ and $x_{a_2}^0$, the coördinate time interval between the reflections at P_1 and P_2 will be given by

$$\begin{aligned} dx^o &= 1/2 \left[x_{e_2}^0 + x_{a_2}^0 - x_{e_1}^0 - x_{a_1}^0\right] \\ &= 1/2 \left[x_{e_2}^0 - x_{e_1}^0 + x_{a_2}^0 - x_{a_1}^0\right] \\ &= 1/2 \left[\Delta x_e^0 + \Delta x_a^0\right] \end{aligned} \tag{35}$$

where the right side of this expression contains only quantities measured by the coordinate clock at B . However, we have already shown that

$$dx^0 = ds \sqrt{-g_{00}} \tag{36}$$

where ds is the increment of proper time on the transported clock. Thus, integrating along the path,

$$\Delta x^0 = \int_{path} \frac{ds}{\sqrt{-g_{00}}} = 1/2 \sum (\Delta x_e^0 + \Delta x_a^0)$$

$$= 1/2 \left[x_a^0 \text{ (last) } - x_a^0 \text{ (first) } + x_e^0 \text{ (last) } - x_e^0 \text{ (first)} \right], \tag{37}$$

where the arguments "last" and "first" refer to the last and first synchronization signals sent out from B . Then since x_a^0 (last) = x_e^0 (last) = x_{arr}^0 , the arrival time of the transported clock at B , we have

$$\Delta x^0 = x_{arr}^0 - 1/2(x_a^0 \text{ (first) } + x_e^0 \text{ (first)})$$

$$= x_{arr}^0 - x_B^0 = x_{arr}^0 - x_A^0$$

$$= \int_{path} \frac{ds}{\sqrt{-g_{00}}} \tag{38}$$

Thus by either method, a consistent set of mutually synchronized coordinate clocks is obtained. This synchronization scheme has the following properties:

1) Reflexivity. If A is synchronized with B , B is synchronized with A. This follows from time-independence of the metric and the fact that the integral of $d\sigma / \sqrt{-g_{00}}$ is independent of direction along a path.

2) Transitivity. If A is synchronized with B , and B is synchronized with C , then A is synchronized with C.

3) Time-independence, because the metric is static.

4) Path-independence, otherwise a global coordinate time would not exist, contrary to hypothesis.

4. Effect of Gravitational Fields of the Sun and Moon

Since it has been suggested in the literature [1, 4] that the gravitational potentials of the sun and moon play a role in processes of clock synchronization, the purpose of this section is to estimate the magnitude of such effects. In the neighborhood of the earth, the gravitational potentials of the sun and moon are small, so it is clear that we can work in a linear approximation in which squares and products of $h_{\mu\nu}$ can be neglected. Consider the term $h_{\mu\nu}$ in a heliocentric coordinate system, and consider first of all the effect of the (static) gravitational potential of the sun on the earth-moon system. We can think of the center of mass of the earth-moon system as freely falling towards the sun. Consequently, at this point we may erect a coordinate system whose origin moves along with the center of mass of the earth-moon system and whose orthonormal axes are obtained by parallel transport of some initially chosen orthonormal tetrad of basis vectors. (We ignore for the moment the gravitational fields of the earth and moon.) Such a coordinate system is termed a normal fermi system and has been discussed by Manasse and Misner[7]. In normal fermi coordinates x^μ, the metric takes the form (see[7] for derivation)

$$g_{00} = -1 + R_{0\ell 0m} x^\ell x^m \tag{39}$$

$$g_{0i} = 2/3\, R_{0\ell im}\, x^\ell x^m \tag{40}$$

$$g_{ij} = \delta_{ij} + 1/3\, R_{i\ell jm}\, x^\ell x^m \tag{41}$$

where $R_{i\ell jm}$ is the Riemann-Christoffel curvature tensor, evaluated at the origin of the normal fermi system, along the freely falling orbit. The distances x^m are proper distances measured from the origin.

The components of the curvature tensor can, however, be evaluated in the heliocentric system of coordinates. Then in transforming them to normal fermi coordinates, the transformation coefficients will be of the form

$$\frac{\partial x^\mu}{\partial x^{\nu'}} = \delta^\mu_\nu + \text{terms of order } \sqrt{GM_s/c^2R_s} \tag{42}$$

where R_s is the radius of the orbit and M_s is the sun's mass. Since we are retaining only first-order terms, the transformation coefficients are Kronecker delta-functions, and the curvature tensor is an invariant.

To see this in more detail, the typical transformation formula for a coordinate will be of the form

$$x = x' + R_s \cos(\omega t')$$

where ω is the angular velocity of the earth in its orbit. Then

$$\frac{\partial x}{\partial x^{o'}} = \frac{R_s \omega}{c} = \frac{R_s}{c}\,\frac{v}{R_s} = \frac{v}{c}\,.$$

But since the velocity is determined by

$$\frac{GM_s}{R_s^2} = \frac{v^2}{R_s}$$

we obtain

$$\frac{v}{c} \simeq \sqrt{\frac{GM_s}{c^2R_s}}\,.$$

Furthermore, in a linear approximation the curvature tensor may be expressed as

$$R_{\alpha\beta\gamma\delta} = h_{\alpha\delta,\beta\gamma} - h_{\beta\delta,\alpha\gamma} - h_{\alpha\gamma,\beta\delta} + h_{\beta\gamma,\alpha\delta} \tag{43}$$

which thus involves second derivatives of the metric tensor.

The largest terms in h_{00} in normal fermi coordinates will thus be of the form

$$-h'_{00,\ell m} x^{\ell} x^{m} \tag{44}$$

where h'_{00} is calculated in heliocentric coordinates in which [3]

$$h'_{00} = \frac{2GM_s}{c^2 r} \tag{45}$$

where r is the distance from the sun to the observation point, and M_s is the sun's mass. Since $x^1 \sim a_1$, the earth's radius, we have

$$h_{00} \cong \frac{GM_s a_1^2}{c^2 R_s^3} \tag{46}$$

Then using

$$M_s = 1.970 \times 10^{33} \text{ gm},$$

$$R_s = 1.495 \times 10^{10} \text{ cm}$$

$$a_1 = 6.3781 \times 10^{8} \text{ cm}$$

we get

$$h_{00} \cong 4 \times 10^{-17} \tag{47}$$

which is negligibly small when computing elapsed times. This term is not negligible when computing forces on particles--it is responsible for tidal forces.

In a similar manner, all other contributions from the sun's mass to $h_{\mu\nu}$ in normal fermi coordinates may be shown to be negligibly small. This conclusion is valid only in the neighborhood of the earth.

These conclusions may be explained qualitatively as follows. A standard clock at a radius slightly greater than that of the freely falling orbit (which for simplicity is assumed circular) will be blue-shifted in the sun's gravitational field since it is further from the origin. The fractional amount by which it is blue-shifted is $g_s \Delta r/c^2$ where $g_s = GM_s/R_s^2$ is the gravitational field strength due to the sun at the earth's orbit. Thus the fractional blue-shift is $GM_s\, \Delta r/c^2 R_s^2$.

However, since the clock is at a greater radius than one at the origin, its velocity in a heliocentric system will be greater and it will be redshifted more due to time dilation. Let v_0 be the orbital velocity of the earth and $v = v_0\,(1 + \Delta r/R_s)$ be the