Analysis of Slender Reinforced Concrete Frames

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Analysis of Slender Reinforced Concrete Frames

Calcul des cadres en béton armé selon la théorie du 2e ordre

Berechnung von Stahlbetonrahmen nach der Theorie 2. Ordnung

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1. Introduction

This paper outlines a method to determine the maximum load carrying capacity of a plane frame with given cross sections and reinforcements.

The present paper based on an investigation described in [1] differs from other investigations [2], [3], [4], [5], [6], [7], [8] in three respects:

— The frame can have an arbitrary geometry.

— An arbitrary load history can be followed.

— A displacement controlled procedure is used which allows the determination of unstable configurations of the frame.

The two main difficulties in the analysis of slender reinforced concrete frames are due to

— the influence of the displacements on the equilibrium of the frame, producing a "geometrical" non-linearity;

— the non-linear stress-strain-time relations for the materials causing a "material" non-linearity.

The two non-linearities are treated separately as shown schematically in Fig. 1. The geometrical non-linearity is considered in a second order elastic analysis. Given are loads P, bending rigidities EI and axial rigidities EA for all elements of the frame. The moment M, the axial force N and the corresponding strain distribution for all sections are determined. The strain

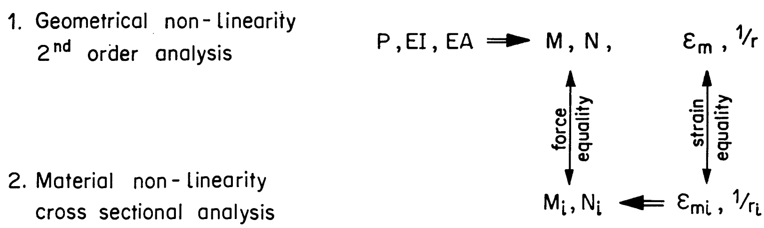

Fig. 1. Schematic illustration of the analysis.

distribution is given by two parameters. Herein, the middle strain ϵ_m in the reference axis of the members and the curvature $1/r$ are used.

The material non-linearity is taken into account in the cross sectional analysis. Given are cross section, reinforcement, stress-strain-time relation for the materials and a strain distribution (ϵ_{mi}, $1/r_i$). The subscript "i" is used for reference to the cross sectional analysis. The moment M_i and axial force N_i are determined.

The elastic and the cross sectional analysis are coupled together by the requirement of equality of the determined forces in the elastic and the cross sectional analysis. Similarly, equality of the strains determined in the elastic analysis and of the strains assumed in the cross sectional analysis must be satisfied.

The critical load of the structure corresponds to the peak on a load-deflection curve separating the stable from the unstable equilibrium configuration. In this range a deformation controlled procedure must be used to assure convergence. Hence, the deformation at some point of the structure is increased by steps to obtain the load-displacement response.

2. Second Order Elastic Analysis

The elastic frame analysis is performed by means of the finite element method. A frame may be visualized as an assemblage of elements interconnected at their ends which are referred to as nodal points or nodes. If the force-displacement relations for each element are known, the equilibrium configuration of the complete structure can be expressed in terms of the nodal

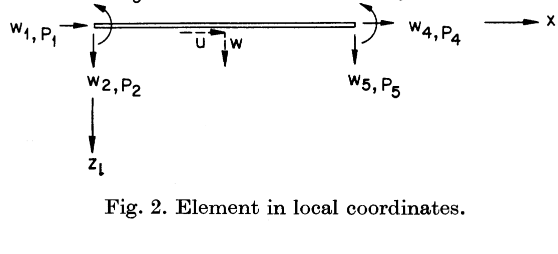

Fig. 2. Element in local coordinates.

displacements. The force-displacement relationship for the element shown in Fig. 2 can be written as

$$[K]\{w\} = \{P\}, \tag{1}$$

in which $\{w\}$ is the displacement vector of the element and $\{P\}$ the corresponding force vector:

$$\{w\} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{Bmatrix}, \qquad \{P\} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix}.$$

Applying standard finite element techniques, the stiffness matrix $[K]$ can be written as

$$[K] = [K_1] + [K_2],$$

where $[K_1]$ is the first order stiffness matrix,
$[K_2]$ is the non-linear geometrical stiffness matrix,
$[K_1]$ and $[K_2]$ are given in Fig. 3.

If the element is inclined at an angle θ with the x-axis, as shown in Fig. 4, the given stiffness matrix above relates to the local coordinates $x_l - z_l$. The

$$[K_1] = \begin{bmatrix} \frac{EA}{l} & & & -\frac{EA}{l} & & \\ & \frac{12EI}{l^3} & \frac{6EI}{l^2} & & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ & & \frac{4EI}{l} & & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ & \text{symmetric} & & \frac{EA}{l} & & \\ & & & & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ & & & & & \frac{4EI}{l} \end{bmatrix} \qquad [K_2] = N \begin{bmatrix} & & & & & \\ & \frac{6}{5l} & \frac{1}{10} & & -\frac{6}{5l} & \frac{1}{10} \\ & & \frac{2l}{15} & & -\frac{1}{10} & -\frac{l}{30} \\ & \text{symmetric} & & & & \\ & & & & \frac{6}{5l} & -\frac{1}{10} \\ & & & & & \frac{2l}{15} \end{bmatrix}$$

Fig. 3. Local element stiffness matrix $[K] = [K_1] + [K_2]$.

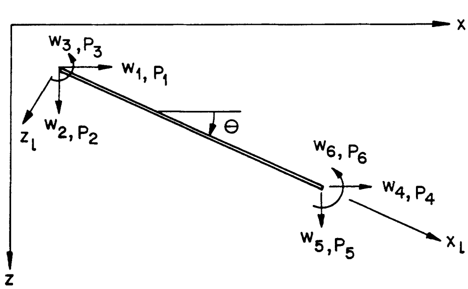

Fig. 4. Global forces and displacements.

global stiffness matrix $[K]$ in the x-z coordinate system is then given by

$$[K] = [R]^T [K_l] [R].$$

$[K_l]$ is the local stiffness matrix given in Fig. 3.

$[R]$ is the transformation matrix relating local displacements $\{w_l\}$ and global displacements $\{w\}$, or local loads $\{P_l\}$ and global loads $\{P\}$ as follows:

$$\{w_l\} = [R]\{w\}, \tag{2}$$

$$\{P_l\} = [R]\{P\},$$

$[R]$ is given in Fig. 5.

The global element stiffness matrix $[K] = [K_1]+[K_2]$ is given in Fig. 6.

$[R] =$

C	S				
-S	C				
		1			
			C	S	
			-S	C	
					1

$\{w_l\} = [R]\,\{w\}$

$\{P_l\} = [R]\,\{P\}$

$S = \sin\Theta$

$C = \cos\Theta$

Fig. 5. Transformation matrix $[R]$.

$[K_1] =$

$\frac{EA}{l} C^2 + \frac{12EI}{l^3} S^2$	$\left(\frac{EA}{l} - 12\frac{EI}{l^3}\right) SC$	$-\frac{6EI}{l^2} S$	$-k_{11}$	$-k_{12}$	k_{13}
	$\frac{EA}{l} S^2 + \frac{12EI}{l^3} C^2$	$\frac{6EI}{l^2} C$	$-k_{12}$	$-k_{22}$	k_{23}
		$4\frac{EI}{l}$	$-k_{13}$	$-k_{23}$	$\frac{2EI}{l}$
	symmetric		k_{11}	k_{12}	$-k_{13}$
				k_{22}	$-k_{23}$
					k_{33}

$[K_2] = N$

$\frac{6}{5\cdot l} S^2$	$-\frac{6}{5l} SC$	$-\frac{1}{10} S$	$-k_{11}$	$-k_{12}$	k_{13}
	$\frac{6}{5l} C^2$	$\frac{1}{10} C$	$-k_{12}$	$-k_{22}$	k_{23}
		$\frac{2l}{15}$	$-k_{13}$	$-k_{23}$	$-\frac{l}{30}$
	symmetric		k_{11}	k_{12}	$-k_{13}$
				k_{22}	$-k_{23}$
					k_{33}

$S = \sin\Theta \quad C = \cos\Theta$

Fig. 6. Global element stiffness matrix $[K] = [K_1]+[K_2]$.

Similar to the force-displacement relationship for the element the force-displacement relationship for the complete structure, or the complete system of elements, can be written as

$$[K]\{w\} = \{P\}, \tag{3}$$

in which $\{w\}$ now contains all nodal displacements and $\{P\}$ all nodal loads.

The system stiffness matrix for the complete structure is obtained by superposition of the individual element stiffness matrices.

When the system stiffness matrix $[K]$ and the load matrix $\{P\}$ have been established, the system of equations is adjusted according to the given boundary conditions. If some displacement, for instance w_j, is identical to zero, this can be taken into account in a simple manner by replacing the diagonal stiffness coefficient K_{jj} by a large number, say 10^{50}.

The solution of the linear system of equations, Eq. (3) is most efficiently carried out taking into account the symmetry and the banded structure of the system stiffness matrix.

It should be noted that the axial force N must be known in order to evaluate the element matrix $[K_2]$ in Fig. 6. The axial force is usually not known in advance, and an iterative procedure must be used. In the first cycle N is chosen equal to zero and the first order forces are calculated. In the second cycle the axial forces found in the first cycle are used.

Usually the axial forces are practically not influenced by the second order effects, such that two cycles are generally sufficient.

When the displacements have been determined, the element forces are found by substituting $\{w\}$ back into Eq. (1). It should be noted that $[K]$ in Eq. (1) is the local element stiffness matrix given in Fig. 3. The global displacements are transformed into local displacements according to Eq. (2).

3. Cross Sectional Analysis

In the cross sectional analysis each section is divided into narrow strips which are assumed to behave as concentrically loaded fibers.

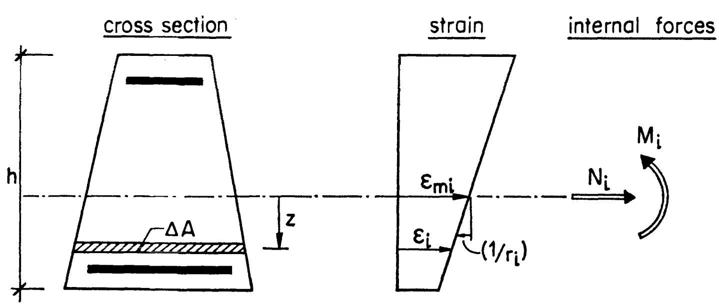

Fig. 7. Cross section, strain distribution and forces.

The resultant forces M_i and N_i are given by (Fig. 7)

$$\begin{aligned} M_i &= \sum \sigma z \Delta A, \\ N_i &= \sum \sigma \Delta A. \end{aligned} \tag{4}$$

The strain ϵ, positive when in tension, is assumed to be linearly distributed over the section. Then

$$\epsilon_i = \epsilon_{mi} - \left(\frac{1}{r_i}\right) z, \tag{5}$$

where ϵ_{mi} is the axial strain in the reference axis,
$\left(\frac{1}{r_i}\right)$ is the curvature,
z is the distance from the axis.

The assumed stress-strain relationship for a virgin concrete specimen (previously not loaded) under instantaneous loading up to failure is shown in Fig. 8. For instantaneous unloading or reloading a linear relation between stress and strain is assumed both for steel and concrete:

$$\sigma = E(\epsilon_i - \epsilon_c - \epsilon_p), \tag{6}$$

where σ is the stress in the considered strip,
E is the "elastic" modulus for the material,
ϵ_i is the total strain,
ϵ_c is the "plastic" strain in the strip from the previous load history,
ϵ_p is the initial strain in the strip, for instance due to prestressing.

The "plastic" strain ϵ_c is due to yielding, creep and shrinkage. At a given time, the magnitude of the plastic strain can be determined from Eq. (6):

$$\epsilon_c = \epsilon - \sigma/E - \epsilon_p. \tag{7}$$

Steel is assumed to be elasto-plastic as shown in Fig. 8. Thus, the stress given by Eq. (6) is limited by the yield stress f_s.

The concrete stress determined from Eq. (6) is assumed to be limited by the stress-strain relationship for a virgin concrete under instantaneous loading. Hence, the stress-strain relationship for a virgin specimen is the envelope curve for the concrete stress-strain relations. The concrete is assumed to have no tensile strength. Concrete shows a time-dependent increase of the plastic strain ϵ_{cc} due to shrinkage and creep.

At a constant sustained stress the plastic strain due to creep is assumed to be given by

$$\epsilon_{cc} = \epsilon_0 \varphi, \tag{8}$$

$$\varphi = \varphi_\infty \frac{t}{T+t}, \tag{9}$$

where φ_∞ is the limiting creep factor,
T is the elapsed time until half the limiting creep is reached,
ϵ_0 is the short-time strain given by:

$$\epsilon_0 = -0.002\left(1-\sqrt{1+\frac{\sigma_c}{f_c}}\right). \tag{10}$$

The hyperbolic expression in Eq. (9) has been used in a number of investigations and seems to be in reasonable agreement with experimental data.

Creep under variable stresses is calculated by dividing the stress history

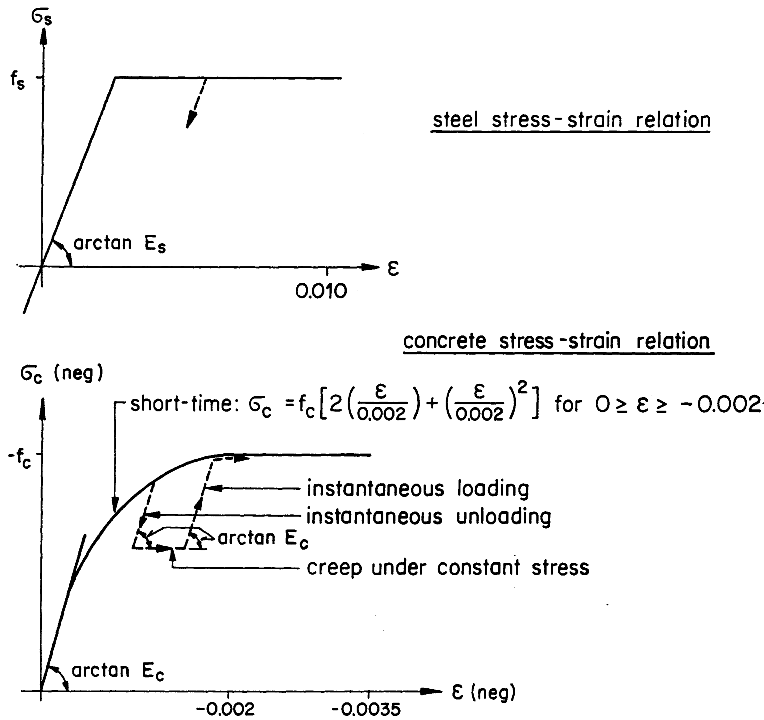

Fig. 8. Stress-strain relations.

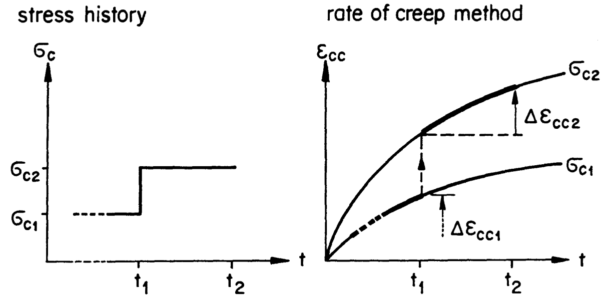

Fig. 9. Creep under variable stresses.

into time intervals and assuming a constant stress within each interval as indicated in Fig. 9. The "rate of creep" method is applied. It is assumed that the creep under variable stresses can be obtained from creep curves for constant stresses. Such curves, for two stress levels σ_{c1} and σ_{c2}, are indicated with solid lines in Fig. 9. In the time interval $\Delta t = t_2 - t_1$ the increase of strain under the stress σ_{c2} is given by

$$\Delta \epsilon_{cc} = \epsilon_0 \varphi_\infty \frac{t_1 + \Delta t}{T + t_1 + \Delta t} - \frac{t_1}{T + t_1}, \tag{11}$$

where

$$\epsilon_0 = -0.002 \left(1 - \sqrt{1 + \frac{\sigma_{c2}}{f_c}}\right).$$

The procedure for determining the stress σ_{c2} at the end of the time interval $t + \Delta t$ is outlined in Fig. 10. Given are the total strain ϵ at the time $t + \Delta t$, the prior plastic strain ϵ_{cc}, the initial strain ϵ_p and the stress σ_{c1} at the time t. As an approximative solution the stress σ_{c1} is used to determine the increase of creep strain $\Delta \epsilon_{cc}$ from Eq. (11). In the case of shrinkage, the corresponding shrinkage strain is added to $\Delta \epsilon_{cc}$. The stress σ_{c2} is determined from Eq. (6). If σ_{c2} exceeds the short-time stress corresponding to the total strain ϵ, the latter stress is chosen. The plastic strain ϵ_{cc} at the time $t + \Delta t$ is given by Eq. (7).

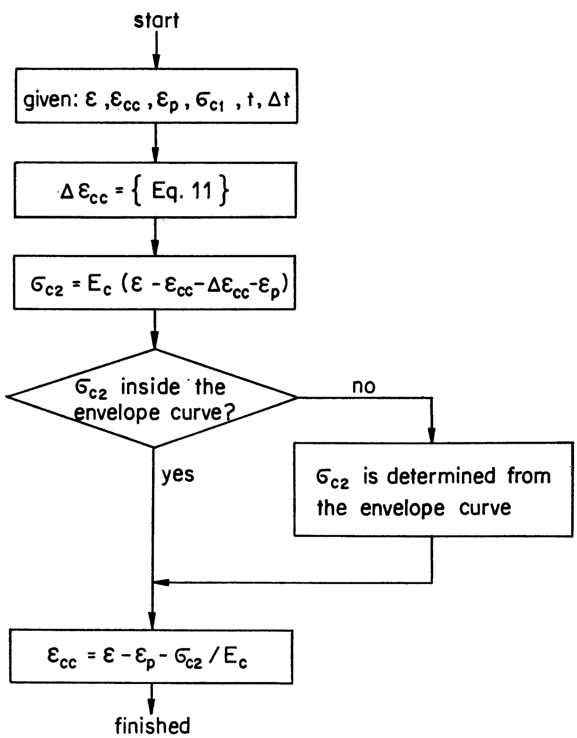

Fig. 10. Determination of stress and "plastic" strain.