APPLICATIONS
OF POTENTIAL THEORY
IN MECHANICS

Selection of new results

V.I. Fabrikant

To my parents

Моим родителям посвящается

PREFACE

It is not easy to find something new in mathematics. It is difficult to find something new in something very old, like the Potential Theory, which was studied by the greatest scientists in the past centuries. And it is extremely difficult to make this find on an elementary level, with no mathematical apparatus involved which would be considered new even in the times of, say, Poisson. This is exactly what the author claims to have done in this book: a new and elementary method is described for solving mixed boundary value problems, and their applications in engineering. The method can solve *non-axisymmetric problems* as easily as axisymmetric ones, *exactly and in closed form*. It enables us to treat *analytically* non-classical domains. The major achievements of the method comprise the derivation of explicit and elementary expressions for the Green's functions, related to a penny-shaped crack and a circular punch, development of the Saint-Venant type *theory of contact and crack problems for general domains*, and investigation of various *interactions* between cracks, punches, and external loadings. The method also provides, as a bonus, a tool for exact evaluation of various two-dimensional integrals involving distances between two or more points. It is believed that majority of these results are beyond the reach of existing methods.

For over twenty years, since being a graduate student in Moscow, the author had been perplexed by an inconsistency between various solutions to the problems in Potential Theory and the way those solutions have been obtained, namely, that the solution was quite elementary, while the apparatus used was very complicated, involving various integral transforms or special functions expansions, which are beyond the comprehension of an ordinary engineer. The author's search for a new method was based on the conviction that an elementary result should be obtainable by elementary means. The method has been found and is presented here in detail. One may just wonder why the method was not discovered at least a century ago.

The book is addressed to a wide audience ranging from engineers, involved in elastic stress analysis, to mathematical physicists and pure mathematicians. While an engineer can find in the book some elementary, ready to use formulae for solving various practical problems, a mathematical physicist might become interested in new applications of the mathematical apparatus presented, and a pure mathematician might interpret some of the results in terms of fractional calculus, investigate the group properties of the operators used, or, having noticed the fact that no attention is paid in the book to a rigorous foundation of the method, might wish to remedy the situation. Due to the mathematical analogy between mixed boundary value problems in elasticity and in other branches of engineering science, *the book should be of interest to specialists in electromagnetics, acoustics, diffusion, fluid mechanics*, etc. Though several such applications have been published by the author, the space considerations did not allow us to include them, but references are given at appropriate places in the book.

The book is *accessible to anyone* with a background in university undergraduate calculus, but should be of interest to established scientists as well. Though the method is elementary, the transformations involved are sometimes very non-trivial and cumbersome, while the final result is usually very simple. The reader who is interested only in application of the general results to his/hers particular problems may skip the long derivations and use the final formulae which require little effort. The reader who wants to master the method in order to solve new problems has to repeat the derivations which are given in sufficient detail. The exercises are important in this regard. They vary from very simple to quite difficult. Some can be used as a subject for a graduate degree thesis.

The book is based entirely on the author's results, and this is why the work of other scientists is mentioned only when such a quotation is inevitable for some reason, like numerical data needed to verify the accuracy of approximate results, comparison with existing results, or pointing out some errors in publications. There are several books and review articles presenting an adequate account of the state-of-the-art in the field. Appropriate references are given for the reader's convenience. The purpose of this book was neither to repeat nor to compete with them.

The development of the method can by no means be considered completed, this is just a beginning. The results presented in the book may be compared to the tip of an iceberg, taking into consideration numerous applications which are still to come. The solution of fundamental problems in a simple form enables us to consider various more complicated problems which were not even attempted before. The *method can be expanded* to spherical, toroidal and other systems of coordinates, so that more complex geometries may benefit from it. The method proved useful in the generalized potential theory as well. Some of these results, though already published by the author, could not be included in the book due to severe restrictions on the book volume.

For the reader's convenience, it was attempted to make each chapter (and section, wherever possible) self-contained. The reader can skip several sections and continue reading, without loosing the ability to understand material. On the other hand, this resulted in repetition of some definitions and descriptions. The author thinks that the additional convenience is worth several extra pages in the book.

The author is grateful to Professor J.R. Barber from Michigan, Professor B. Noble from England, and Professor J.R. Rice from Harvard who agreed to read the manuscript and expressed their opinion.

The book contains so much new material that some misprints and errors are inevitable, though every effort was made to eliminate them. The author would be grateful for every communication in this regard. All the readers' comments are welcome. The address is

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CONTENTS

INTRODUCTION 7

CHAPTER 1. DESCRIPTION OF THE NEW METHOD

1.1 Integral representation for the reciprocal of the distance between two points 15
1.2 Properties of the $\mathcal{L}$-operators 22
1.3 Some further integral representations 25
1.4 Internal mixed boundary value problem for a half-space 31
1.5 External mixed boundary value problem for a half-space 43
1.6 Some fundamental integrals 63

CHAPTER 2. MIXED BOUNDARY VALUE PROBLEMS IN ELASTICITY

2.1 General solution 73
2.2 Point force solutions 77
2.3 Internal mixed problem of type I 81
2.4 External mixed problem of type I 92
2.5 Integral representation for q^2/R^3 101
2.6 Internal mixed problem of type II 105
2.7 External mixed problem of type II 111
2.8 Inverse crack problem in elasticity 129

CHAPTER 3. MIXED-MIXED BOUNDARY VALUE PROBLEMS

3.1 General formulation of the problem 142
3.2 Internal axisymmetric mixed-mixed problem 145
3.3 External axisymmetric mixed-mixed problem 171
3.4 Generalization for non-homogeneous half-space 177
3.5 Effect of a shearing force and a tilting moment on a bonded circular punch 182
3.6 Non-axisymmetric internal mixed-mixed problem 188
3.7 Non-axisymmetric external mixed-mixed problem 200
Appendix A3.1 207
Appendix A3.2 211
Appendix A3.3 215

CHAPTER 4. APPLICATIONS IN FRACTURE MECHANICS
4.1 Flat crack under arbitrary normal loading 220
4.2 Point force loading of a penny-shaped crack 227
4.3 Concentrated load outside a circular crack 229
4.4 Plane crack under arbitrary shear loading 231
4.5 Penny-shaped crack under uniform pressure 243
4.6 Penny-shaped crack under uniform shear loading 248
4.7 Asymptotic behavior of stresses and displacements near the crack rim 252
4.8 Flat crack of general shape 259
4.9 General crack under uniform shear 273
4.10 Close interaction of pressurized coplanar circular cracks 275
4.11 Close interaction of coplanar circular cracks under shear loading 292
Appendix A4.1 307
Appendix A4.2 311
Appendix A4.3 312
Appendix A4.4 317

CHAPTER 5. APPLICATION TO CONTACT PROBLEMS
5.1 Contact problem for a smooth punch. 324
5.2 Flat centrally loaded circular punch 328
5.3 Inclined circular punch on an elastic half-space 332
5.4 Flat punch of arbitrary planform under the action of a normal centrally applied force 336
5.5 Inclined flat punch of general shape 347
5.6 Curved punch of general planform 367
5.7 Flat flexible punch of general planform under shifting load 387
5.8 Reissner-Sagoci problem for general domains 393
5.9 Interaction between punches subjected to normal pressure 402
5.10 Interaction between flexible punches under shifting loading 412
5.11 Contact problem for a rough punch 422
Appendix A5.1 425

REFERENCES 439

SUBJECT INDEX 447

INTRODUCTION

A short survey of methods for solving mixed boundary value problems of potential theory is given, and some of their limitations are pointed out. The necessity for developing a new method is justified. A concise description of each chapter is given for the reader's convenience.

Various applications of potential theory in electrostatics, heat transfer, elasticity, diffusion, and other branches of engineering science are well known, and have attracted significant attention of scientists like Laplace, Poisson, Green, Beltrami, Kirchhoff, Lord Kelvin, Hobson, and others who made a significant contribution to the field during past centuries. The boundary value problems with mixed conditions are the most difficult to solve, and, at the same time, they are among the most important in various engineering applications. Recall the celebrated problem of an electrified circular disc. It is next to impossible even to name every scientist who solved the problem by one method or another. We can specify two categories of methods. The first one requires construction of the Green's function, after which each particular solution can be presented in quadratures. The second category encompasses various integral transform methods. Let us briefly discuss both categories.

Hobson (1899) has constructed the Green's function for a circular disc and a spherical bowl using a method due to Sommerfeld. He has used toroidal coordinates τ, σ, ϕ (in our notation), which are related to the cartesian x, y, and z as follows:

$$x = \frac{a\sinh\tau \, \cos\phi}{\cosh\tau - \cos\sigma}\,, \qquad y = \frac{a\sinh\tau \, \sin\phi}{\cosh\tau - \cos\sigma}\,, \qquad z = \frac{a\sin\sigma}{\cosh\tau - \cos\sigma}\,.$$

Here a is the disc radius. The potential function V at the point (τ_0,σ_0,ϕ_0)

external to the disc, which on the disc takes the values $v(\tau,\phi)$, can be expressed

$$V(\tau_0,\sigma_0,\phi_0) = \frac{1}{\pi^2}\iint_S \left\{ \frac{(1 + \cosh\tau)\cos(\sigma_0/2)}{2aR[\cosh^2(\alpha/2) - \sin^2(\sigma_0/2)]^{1/2}} \right.$$

$$\left. + \frac{z}{R^3}\tan^{-1}\frac{(1 - \cos\sigma_0)^{1/2}}{(\cosh\alpha + \cos\sigma_0)^{1/2}} \right\} v(\tau,\phi)\ \mathrm{d}S, \tag{0.1}$$

where R is the distance between the points (τ_0,σ_0,ϕ_0) and (τ,σ,ϕ), and α is defined by $\cosh\alpha = \cosh\tau_0\cosh\tau - \sinh\tau_0\sinh\tau\cos(\phi-\phi_0)$. The practical value of expression (0.1) is quite limited, since there seems to be no way to evaluate the integral involved, even in the simplest case when $v(\tau,\phi)$ is constant. Hobson had to use a very ingenious method in order to find the potential function for v=const and $v=\mu x$, μ=const. On the other hand, it turns out that the integral in (0.1) is computable in elementary functions for any polynomial v (in Cartesian coordinates). This was one of the reasons that prompted the author to look for an alternative approach, which would be as general as (0.1), and, on the other hand, would allow elementary and straightforward computation of the integrals involved.

Consideration of the mathematically equivalent problem of a circular punch pressed against an elastic half-space leads to the integral equation (Galin, 1953)

$$\omega(\rho,\phi) = H\int_0^{2\pi}\int_0^{a} \frac{\sigma(\rho_0,\phi_0)}{R}\ \rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0. \tag{0.2}$$

Here a is the punch radius, H is an elastic constant, ω is the normal displacement under the punch (a known function), σ stands for the pressure exerted by the punch (an unknown function), and R is the distance between the points (ρ,ϕ) and (ρ_0,ϕ_0). Leonov (1953) has obtained a closed form exact solution of integral equation (0.2) by a very ingenious method. His result reads in our notation

$$\sigma(\rho,\phi) = -\frac{1}{4\pi^2 H}\left\{\Delta\int_0^{2\pi}\int_0^{a} \frac{\omega(\rho_0,\phi_0)}{R}\ \rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0\right.$$

$$+ \ \frac{2}{\pi} \int_0^{2\pi} \int_0^a \left[\frac{R}{\eta} \ - \ \tan^{-1}(\frac{R}{\eta})\right] \frac{\omega(\rho_0,\phi_0)}{R^3} \ \rho_0 d\rho_0 d\phi_0 \Bigg\}. \qquad (0.3)$$

Here $\eta=[(a^2-\rho^2)(a^2-\rho_0^2)]^{1/2}/a$ and Δ is the two-dimensional Laplace operator. One can observe the same handicap in (0.3): difficulty to evaluate the integrals directly, even in the simplest case when ω is constant. Again, it is clear that the integrals are computable in elementary functions for any prescribed polynomial displacement. This indicates a gap in our knowledge which needs to be filled.

The integral transform method, involving dual integral equations, was originated, probably by Weber (1873) and Beltrami (1881), and continued by Busbridge (1938) and others. Significant achievements in the systematic application of the method to various problems belong to Sneddon. The reader is referred to the books by Sneddon (1951, 1966) and Ufliand (1967) for additional references. Some quite remarkable results were obtained by Ufliand (1977). Despite this success, it has always been the author's conviction that our use of integral transforms generally indicates our inability to solve problems directly. To this end, two illustrative examples are presented.

Here is how the problem of a circular disc, charged to a potential v_0=const, is solved by the dual integral equation method. It is necessary to find a harmonic function V, vanishing at infinity, and subjected to the mixed boundary conditions on the plane z=0:

$$V \ = \ v_0, \ \text{for} \ \rho\leq a; \qquad \frac{\partial V}{\partial z} \ = \ 0, \ \text{for} \ \rho>a. \qquad (0.4)$$

The solution is presented in the form

$$V(\rho,z) \ = \ \int_0^\infty A(t)e^{-tz}J_0(t\rho) \ \frac{dt}{t} \ . \qquad (0.5)$$

Here J_0 is the Bessel function of zero order, and $A(t)$ is the as yet unknown function which should be chosen to satisfy (0.4). Substitution of the boundary conditions (0.4) in (0.5) leads to the dual integral equations

$$\int_0^\infty A(t)J_0(t\rho)\,\frac{\mathrm{d}t}{t} = v_0, \quad \text{for} \quad 0\le\rho\le a;$$

$$\int_0^\infty A(t)J_0(t\rho)\mathrm{d}t = 0, \quad \text{for} \quad \rho>a. \tag{0.6}$$

By using the discontinuous Weber-Schafheitlin integrals, one can deduce that

$$A(t) = \frac{2}{\pi}\, v_0 \sin(at). \tag{0.7}$$

The solution (0.5) can now be rewritten as

$$V(\rho,z) = \frac{2}{\pi}\, v_0 \int_0^\infty \mathrm{e}^{-tz}\sin(at)\; J_0(t\rho)\,\frac{\mathrm{d}t}{t}\;, \quad \text{for} \quad z\ge 0, \tag{0.8}$$

and the charge density σ over the disc is given by

$$\sigma(\rho) = \frac{v_0}{\pi^2}\int_0^\infty J_0(t\rho)\;\sin(at)\;\mathrm{d}t. \tag{0.9}$$

Both integrals (0.8) and (0.9) are computable in elementary functions. For example, the last integral yields

$$\sigma(\rho) = \frac{v_0}{\pi^2(a^2 - \rho^2)^{1/2}}. \tag{0.10}$$

There seems to be a discrepancy between the simplicity of the final result and the apparatus used to obtain it. The general idea, that an elementary result should be obtained by elementary means, calls for a search for a new and elementary approach.

The second example comes from consideration of the simplest case of a penny-shaped crack of radius a, subjected to axisymmetric pressure $p(\rho)$. The corresponding potential function f can be found in (Kassir and Sih, 1975) as follows:

$$f(\rho,z) = \int_0^\infty A(s)\ J_0(\rho s)\ \exp(-sz)\ \frac{ds}{s}\ , \tag{0.11}$$

where

$$A(s) = -\frac{1}{\pi\mu}\int_0^a \sin st\ dt \int_0^t \frac{rp(r)\ dr}{(t^2 - r^2)^{1/2}}\ . \tag{0.12}$$

The user has to substitute the explicit expression for p in (0.12), and to evaluate two consecutive integrals. The result is to be substituted in (0.11), and an infinite integral with Bessel function is to be evaluated. It seems natural to try to spare one integration by substituting (0.12) in (0.11), changing the order of integration and evaluating the integral (Gradshtein and Ryzhik, 1963, formula 6.752.1):

$$\int_0^\infty \sin st\ J_0(\rho s)\ \exp(-sz)\ \frac{ds}{s} = \sin^{-1}\frac{t}{l_2(t)}\ , \tag{0.13}$$

where the notation

$$l_2(t) = \frac{1}{2}\{[(\rho + t)^2 + z^2]^{1/2} + [(\rho - t)^2 + z^2]^{1/2}\}. \tag{0.14}$$

was introduced. Formulae (0.11) and (0.12) can be combined to give

$$f(\rho,z) = -\frac{1}{\pi\mu}\int_0^a \sin^{-1}\frac{t}{l_2(t)}\ dt \int_0^t \frac{p(\rho_0)\rho_0 d\rho_0}{(t^2 - \rho_0^2)^{1/2}}\ . \tag{0.15}$$

Note that expression (0.15) contains no trace of the integral transform, and thus gives us a hint that a direct and elementary solution is indeed possible. For example, in the case of a uniform loading, p=const., and the potential function (0.15) will take the form:

$$f(\rho,z) = -\frac{p}{\pi\mu}\int_0^a t\ \sin^{-1}\frac{t}{l_2(t)}\ dt. \tag{0.16}$$

The integral in (0.16), though looking formidable due to (0.14), can be evaluated in elementary functions, namely,

$$f(\rho,z) = -\frac{p}{4\pi\mu}\left[(2a^2 + 2z^2 - \rho^2)\sin^{-1}\left(\frac{a}{l_2}\right) + l_1(\rho^2 - l_1^2)^{1/2} - 2z(a^2 - l_1^2)^{1/2}\right]. \tag{0.17}$$

Here the abbreviations l_1 and l_2 stand for $l_1(a)$ and $l_2(a)$ respectively, with l_2 defined by (0.14), and

$$l_1(t) = \frac{1}{2}\{[(\rho + t)^2 + z^2]^{1/2} - [(\rho - t)^2 + z^2]^{1/2}\}. \tag{0.18}$$

One can show that arbitrary polynomial loading in (0.15) will lead to an elementary expression for the potential function, and thus to an elementary complete solution.

The situation can now be summarized. The Green's function approach is the most general, the main impediment being the inability of direct derivation of results which were usually *constructed* due to some ingenious considerations. By contrast, the integral transform method allows a straightforward derivation of the results, but it is the least general, since each particular problem has to be solved from beginning to the end. The method is best suited to axisymmetric problems. In the general case, separate solutions have to be obtained for each harmonic. Non-axisymmetric problems involving various interactions (several *arbitrarily located* charged discs, interaction of punches and cracks, etc.) are extremely difficult to solve by the integral transform method. A new method has to be found which would be as general as the Green's function method, and, at the same time, it has to be elementary and straightforward, with no integral transforms or special function expansions involved.

When one problem has been solved by a complicated method, it is often possible to find another method to solve the same problem more simply. The new method would have been of little value if all it could do were to solve more easily the already solved problems. The main advantage of the new method presented here is its ability to solve non-axisymmetric problems as easily as axisymmetric ones, which in turn opens up new horizons, and allows us to solve some problems which were not even considered before, namely, the analytical treatment of nonclassical domains, and the solution to various interaction problems.

The general description of the method is given in Chapter 1. It starts with a derivation of the basic integral representation for the reciprocal of the distance between two points, followed by several generalizations. A closed form exact solution is given to the non-axisymmetric mixed problem of potential theory for a half-space, with Dirichlet conditions prescribed inside a circle, and Neumann conditions given on the outside, and vice-versa. Some integrals, which are of fundamental value to the method, are evaluated in elementary functions.

Chapter 2 is devoted to the mixed boundary value problems of an elastic half-space. The general solution is expressed in terms of three harmonic functions. A classification of internal and external mixed problems of type I and type II is introduced. The problems of the first type are characterized by mixed conditions with respect to normal parameters (the pressure and normal displacement), with the shear stress prescribed all over the boundary. These problems are solved for a non-homogeneous half-space, with the elasticity modulus assumed to be a power function of the depth. The case when the boundary conditions are mixed with respect to tangential displacements and stresses, with the normal stress being prescribed all over the boundary, is classified as the type II problem. Each type is considered separately. Exact closed form solution and various examples of punch and crack problems are presented.

The problem is called mixed-mixed when the boundary conditions are mixed with respect to both normal and tangential parameters. This kind of problem is the most difficult to solve due to the coupling of the governing integral equations, which can no longer be solved separately. The axisymmetric and non-axisymmetric internal and external problems are considered in Chapter 3. The exact solution is given in terms of Fourier series expansions. A flat punch, bonded to a transversely isotropic elastic half-space and subjected to general loading is considered in detail. The interaction of exterior loading with the punch is also investigated.

While the problems solved in Chapter 2 deal primarily with the stresses and displacements in the plane $z=0$, the *complete* solution to various crack problems is the subject of Chapter 4. The solution is called *complete*, when explicit expressions for the field of stresses and displacements is defined in the whole space. The cases of a penny-shaped crack under arbitrary normal and tangential loadings are considered separately. Explicit expressions are derived for all the Green’s functions involved. All the results are given in terms of elementary functions. These solutions enable us to solve more complicated problems of interaction of a penny-shaped crack and an exterior load. A set of non-singular governing integral equations is derived for the interaction between coplanar circular cracks, subjected to normal pressure and shear loading. An approximate analytical solution is presented for the case of a flat crack of general shape, subjected to a uniform pressure or a uniform shear. The solution is exact for an ellipse, and is expected to be satisfactory for a wide variety of shapes. A comparison is made with various numerical results, available in the literature, and a very good accuracy is established in many cases. The method is less accurate for domains with sharp angles and the aspect ratio far away from unity.

A general solution in terms of one harmonic function is given in Chapter 5 to the problem of a smooth punch, penetrating a transversely isotropic elastic half-space. The main potential function and all the Green's functions are expressed in terms of elementary functions for a circular punch of arbitrary profile. It is shown that the complete solution is also presentable in elementary functions for a general polynomial profile. The knowledge of a complete solution, combined with the reciprocal theorem, enables us to solve more complicated problems of interaction of exterior loadings with punches, and the interaction between punches. The general method is applied to the analytical treatment of nonclassical contact problems. Again, the solution is exact for an elliptical punch, and is expected to be satisfactory for a punch of arbitrary planform. The cases of a flat centrally and noncentrally loaded punch, and the case of a curved punch are considered in detail. An extensive comparison is made with the numerical results, available in the literature. As it was in the case of crack problems, the agreement is quite satisfactory, except for the domains with sharp angles and the aspect ratio far away from unity. An analytical solution is given to the problem of a non-smooth punch, subjected to normal and shear loading, with the Coulomb friction law assumed between the punch base and the elastic half-space.

The best way to master a new method is through the exercises. Each chapter contains a certain number of them, the majority of exercises are supplied with an answer, a hint or a complete solution. The reader is encouraged to do them all. The transformations involved are elementary, though sometimes very non-trivial, and require some ingenuity.

CHAPTER 1

DESCRIPTION OF THE NEW METHOD

Some integral representations, which are of fundamental value to the method, are derived. An exact closed form solution is given to the mixed boundary value problem of potential theory for a half-space, with a circular line of division of boundary conditions.

1.1 Integral representation for the reciprocal of the distance between two points

The author has decided to start with a derivation of a new integral representation for the reciprocal of the distance between two points located in the plane z=0 since this quantity is very important in potential theory. Here we repeat the derivation leading to such a representation, as it was given in (Fabrikant 1971e). Consider the expression

$$\frac{1}{R^{1+u}} = \frac{1}{(\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0))^{(1+u)/2}}, \tag{1.1.1}$$

where u is a constant and $-1<u<1$. The standard expansion of (1.1.1) in Fourier series will take the form

$$\frac{1}{R^{1+u}} = \sum_{n=-\infty}^{\infty} \frac{e^{in(\phi-\phi_0)}}{2\pi} \int_0^{2\pi} \frac{e^{-in\psi}d\psi}{(\rho^2 + \rho_0^2 - 2\rho\rho_0\cos\psi)^{(1+u)/2}}$$

$$= \sum_{n=-\infty}^{\infty} \frac{e^{in(\phi-\phi_0)}}{2\pi\rho_0^{1+u}} \frac{2\pi\Gamma[n + (1 + u)/2]}{\Gamma[(1 + u)/2]\ \Gamma(n + 1)} \left(\frac{\rho}{\rho_0}\right)^n F\left(\frac{1 + u}{2},\ n + \frac{1 + u}{2},\ n+1;\ \frac{\rho^2}{\rho_0^2}\right). \tag{1.1.2}$$

Here F stands for the Gauss hypergeometric function. By using another integral representation

$$F\left(\frac{1+u}{2},\ n+\frac{1+u}{2},\ n+1;\ z\right)$$

$$= \frac{2\Gamma(n+1)}{\Gamma[n+(1+u)/2]\ \Gamma[1-(1+u)/2]} \int_0^1 \frac{t^{2n+u}(1-t^2)^{-(1+u)/2}}{(1-zt^2)^{(1+u)/2}}\, dt,$$

expression (1.1.2) can be transformed into

$$\frac{1}{R^{1+u}} = \frac{2}{\pi}\cos\frac{\pi u}{2}\sum_{n=-\infty}^{\infty}\frac{e^{in(\phi-\phi_0)}}{(\rho\rho_0)^n}\int_0^{min(\rho_0,\rho)}\frac{x^{2n+u}dx}{\left[(\rho^2-x^2)(\rho_0^2-x^2)\right]^{(1+u)/2}}\ . \qquad (1.1.3)$$

Summation in (1.1.3) finally gives

$$\frac{1}{R^{1+u}} = \frac{1}{(\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0))^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^{min(\rho_0,\rho)}\frac{\lambda\left(\frac{x^2}{\rho\rho_0},\ \phi-\phi_0\right)\ x^u dx}{\left[(\rho^2-x^2)(\rho_0^2-x^2)\right]^{(1+u)/2}}\ . \qquad (1.1.4)$$

Here the notation was introduced

$$\lambda(k,\psi) = \frac{1-k^2}{1+k^2-2k\ \cos\psi}\ . \qquad (1.1.5)$$

After one cumbersome derivation is finished, we can always find a way to do it much simpler. Indeed, if we introduce a new variable

$$\eta(x) = [(\rho^2-x^2)(\rho_0^2-x^2)]^{1/2}/x, \qquad (1.1.6)$$

expression (1.1.4) may be rewritten as

$$\frac{1}{R^{1+u}} = \frac{2}{\pi} \cos \frac{\pi u}{2} \int_0^\infty \frac{\eta^{-u} d\eta}{R^2 + \eta^2}. \tag{1.1.7}$$

The integral in (1.1.7) can be evaluated by using formula (3.241.4) from (Gradshtein and Ryzhik, 1963), thus proving the identity. Note that parameter η will be used throughout the book also for the case when $x>\max(\rho,\rho_0)$, and expression (1.1.6) in this case is interpreted as

$$\eta(x) = (x^2 - \rho^2)^{1/2}(x^2 - \rho_0^2)^{1/2}/x.$$

One can deduce from (1.1.7) that in the particular case when $u=0$, the integral in (1.1.4) can be evaluated as indefinite, and we have a very important representation

$$\int \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0)\, dx}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}} = -\frac{1}{R} \tan^{-1}\left[\frac{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}}{xR}\right]. \tag{1.1.8}$$

All the results above are related to the distance between two points in the plane $z=0$. We need to generalize them to represent

$$\frac{1}{R_0^{1+u}} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{(1+u)/2}} . \tag{1.1.9}$$

One can observe that representation (1.1.4) remains valid if we formally substitute ρ and ρ_0 by arbitrary quantities l_1 and l_2. We need to choose them so that

$$\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2 = l_1^2 + l_2^2 - 2l_1l_2\cos(\phi-\phi_0). \tag{1.1.10}$$

This leads to two equations

$$l_1^2 + l_2^2 = \rho^2 + \rho_0^2 + z^2, \qquad l_1l_2 = \rho\rho_0. \tag{1.1.11}$$

The solution will take the form

$$l_1(\rho_0,\rho,z) = \frac{1}{2}\{[(\rho + \rho_0)^2 + z^2]^{1/2} - [(\rho - \rho_0)^2 + z^2]^{1/2}\}, \tag{1.1.12}$$

$$l_2(\rho_0,\rho,z) = \frac{1}{2}\{[(\rho + \rho_0)^2 + z^2]^{1/2} + [(\rho - \rho_0)^2 + z^2]^{1/2}\}. \qquad (1.1.13)$$

Hereafter the following abbreviations will be used:

$$l_1(x) \equiv l_1(x,\rho,z), \qquad l_2(x) \equiv l_2(x,\rho,z), \qquad (1.1.14)$$

$$l_1 \equiv l_1(a,\rho,z), \qquad l_2 \equiv l_2(a,\rho,z). \qquad (1.1.15)$$

Note the limiting properties

$$\lim_{z\to 0} l_1(x) = \min(x,\rho), \qquad \lim_{z\to 0} l_2(x) = \max(x,\rho). \qquad (1.1.16)$$

In view of the properties above, the representation (1.1.4) can be generalized

$$\frac{1}{R_0^{1+u}} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^{l_1(\rho_0)} \frac{\lambda(\frac{x^2}{\rho\rho_0},\ \phi-\phi_0)\ x^u dx}{\{[l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\}^{(1+u)/2}}\ . \qquad (1.1.17)$$

Formula (1.1.17) simplifies when u=0

$$\frac{1}{R_0} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}}$$

$$= \frac{2}{\pi}\int_0^{l_1(\rho_0)} \frac{\lambda(\frac{x^2}{\rho\rho_0},\ \phi-\phi_0)\ dx}{\{[l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\}^{1/2}}\ . \qquad (1.1.18)$$

Again, one can notice that the integral in (1.1.18) may be evaluated as indefinite

$$\int \frac{\lambda(\frac{x^2}{\rho\rho_0},\ \phi-\phi_0)\ dx}{\{[l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\}^{1/2}} = -\frac{1}{R_0}\tan^{-1}\frac{\{[l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\}^{1/2}}{xR_0}.$$

(1.1.19)

The last representation is very important and will be widely used throughout the book.

Another series of useful formulae can be obtained from those above by a simple change of variables, namely,

$$\int \frac{\lambda\left(\frac{\rho\rho_0}{x^2},\ \phi-\phi_0\right)\ dx}{\{[x^2 - l_1^2(\rho_0)][x^2 - l_2^2(\rho_0)]\}^{1/2}} = \frac{1}{R_0}\ \tan^{-1}\frac{\{[x^2 - l_1^2(\rho_0)][x^2 - l_2^2(\rho_0)]\}^{1/2}}{xR_0},$$

(1.1.20)

$$\frac{1}{R_0^{1+u}} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2}\int_{l_2(\rho_0)}^{\infty}\frac{\lambda\left(\frac{\rho\rho_0}{x^2},\ \phi-\phi_0\right)\ x^u dx}{\{[x^2 - l_1^2(\rho_0)][x^2 - l_2^2(\rho_0)]\}^{(1+u)/2}}\ , \quad (1.1.21)$$

$$\frac{1}{R_0} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}}$$

$$= \frac{2}{\pi}\int_{l_2(\rho_0)}^{\infty}\frac{\lambda\left(\frac{\rho\rho_0}{x^2},\ \phi-\phi_0\right)\ dx}{\{[x^2 - l_1^2(\rho_0)][x^2 - l_2^2(\rho_0)]\}^{1/2}}\ , \quad (1.1.22)$$

$$\int\frac{\lambda\left(\frac{\rho\rho_0}{x^2},\ \phi-\phi_0\right)\ dx}{(x^2 - \rho^2)^{1/2}(x^2 - \rho_0^2)^{1/2}} = \frac{1}{R}\ \tan^{-1}\left[\frac{(x^2 - \rho^2)^{1/2}(x^2 - \rho_0^2)^{1/2}}{xR}\right]. \quad (1.1.23)$$

The representations above are useful for solving external mixed boundary value problems.

Several modifications of (1.1.19) are available. For example, we can write

$$\int \frac{\lambda(\frac{x^2}{\rho\rho_0},\ \phi-\phi_0)\ \mathrm{d}x}{(\rho^2 - x^2)^{1/2}[\rho_0^2 - g^2(x)]^{1/2}} = -\frac{1}{R_0}\tan^{-1}\frac{(\rho^2 - x^2)^{1/2}[\rho_0^2 - g^2(x)]^{1/2}}{xR_0}. \tag{1.1.24}$$

Here

$$g(x) = x[1 + z^2/(\rho^2 - x^2)]^{1/2}. \tag{1.1.25}$$

It is important to notice that the function $g(x)$ is inverse to l_1 for $x^2<\rho^2$, and is inverse to l_2 for $x^2>\rho^2+z^2$. Introduction of a new variable $x=l_1(y)$, $y=g(x)$ transforms (1.1.24) into

$$\int \frac{[l_2^2(y) - y^2]^{1/2}}{(\rho_0^2 - y^2)^{1/2}[l_2^2(y) - l_1^2(y)]}\ \lambda\left(\frac{l_1^2(y)}{\rho\rho_0},\ \phi-\phi_0\right)\mathrm{d}y$$

$$= -\frac{1}{R_0}\tan^{-1}\frac{(\rho_0^2 - y^2)^{1/2}[l_2^2(y) - y^2]^{1/2}}{yR_0} \tag{1.1.26}$$

A particular case of (1.1.18), when z=0, reads

$$\frac{1}{R} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}}$$

$$= \frac{2}{\pi}\int_0^{min(\rho_0,\rho)} \frac{\lambda(\frac{x^2}{\rho\rho_0},\ \phi-\phi_0)\ \mathrm{d}x}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}}. \tag{1.1.27}$$

The same result takes another form due to (1.1.22)

$$\frac{1}{R} = \frac{2}{\pi}\int_{max(\rho_0,\rho)}^{\infty} \frac{\lambda(\frac{\rho\rho_0}{x^2},\ \phi-\phi_0)\ \mathrm{d}x}{(x^2 - \rho^2)^{1/2}(x^2 - \rho_0^2)^{1/2}}. \tag{1.1.28}$$