

# An Algebraic Introduction to Complex Projective Geometry

## 1. Commutative algebra

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Published by the Press Syndicate of the University of Cambridge  
The Pitt Building, Trumpington Street, Cambridge CB2 1RP  
40 West 20th Street, New York, NY 10011-4211, USA  
10 Stamford Road, Melbourne, Victoria 3166, Australia

© Cambridge University Press 1996

First published 1996

Printed in Great Britain at the University Press, Cambridge

*Library of Congress cataloguing in publication data available*

*A catalogue record for this book is available from the British Library*

ISBN 0 521 48072 8 hardback

# Contents

<b>1</b>	<b>Rings, homomorphisms, ideals</b>	<b>1</b>
1.1	Ideals. Quotient rings . . . . .	2
1.2	Operations on ideals . . . . .	6
1.3	Prime ideals and maximal ideals . . . . .	7
1.4	Nilradicals and Jacobson radicals . . . . .	10
1.5	Comaximal ideals . . . . .	11
1.6	Unique factorization domains (UFDs) . . . . .	12
1.7	Exercises . . . . .	14
<b>2</b>	<b>Modules</b>	<b>17</b>
2.1	Submodules. Homomorphisms. Quotient modules . . . . .	18
2.2	Products and direct sums . . . . .	20
2.3	Operations on the submodules of a module . . . . .	21
2.4	Free modules . . . . .	22
2.5	Homomorphism modules . . . . .	24
2.6	Finitely generated modules . . . . .	25
2.7	Exercises . . . . .	28
<b>3</b>	<b>Noetherian rings and modules</b>	<b>29</b>
3.1	Noetherian rings . . . . .	29
3.2	Noetherian UFDs . . . . .	31
3.3	Primary decomposition in Noetherian rings . . . . .	32
3.4	Radical of an ideal in a Noetherian ring . . . . .	33
3.5	Back to primary decomposition in Noetherian rings . . . . .	34
3.6	Minimal prime ideals . . . . .	35
3.7	Noetherian modules . . . . .	36
3.8	Exercises . . . . .	37
<b>4</b>	<b>Artinian rings and modules</b>	<b>39</b>
4.1	Artinian rings . . . . .	39
4.2	Artinian modules . . . . .	43
4.3	Exercises . . . . .	43

<b>5</b>	<b>Finitely generated modules over Noetherian rings</b>	<b>45</b>
5.1	Associated prime ideals . . . . .	45
5.2	Finite length modules . . . . .	48
5.3	Finitely generated modules over principal ideal rings . . . . .	51
5.4	The Artin–Rees lemma and Krull’s theorem . . . . .	55
5.5	Exercises . . . . .	56
<b>6</b>	<b>A first contact with homological algebra</b>	<b>59</b>
6.1	Some abelian categories . . . . .	59
6.2	Exact sequences . . . . .	61
6.3	Tensor products and homomorphism modules . . . . .	65
6.4	Dualizing module on an artinian ring . . . . .	68
6.5	Gorenstein artinian rings . . . . .	73
6.6	Exercises . . . . .	75
<b>7</b>	<b>Fractions</b>	<b>79</b>
7.1	Rings of fractions . . . . .	79
7.2	Fraction modules . . . . .	82
7.3	Support of a module . . . . .	86
7.4	Localization of ideals . . . . .	92
7.5	Localization and UFDs . . . . .	94
7.6	Localization and primary decomposition . . . . .	96
7.7	Back to minimal prime ideals . . . . .	98
7.8	Localization and associated prime ideals . . . . .	99
7.9	Exercises . . . . .	101
<b>8</b>	<b>Integral extensions of rings</b>	<b>103</b>
8.1	Algebraic elements, integral elements . . . . .	103
8.2	Finite extensions, integral extensions . . . . .	105
8.3	Going-up and going-down theorems . . . . .	108
8.4	Exercises . . . . .	112
<b>9</b>	<b>Algebraic extensions of fields</b>	<b>113</b>
9.1	Finite extensions . . . . .	113
9.2	$K$ -isomorphisms in characteristic zero . . . . .	116
9.3	Normal extensions . . . . .	117
9.4	Trace and norm . . . . .	121
9.5	Roots of one and cyclic Galois groups . . . . .	123
9.6	Exercises . . . . .	126
<b>10</b>	<b>Noether’s normalization lemma</b>	<b>129</b>
10.1	Transcendence degree . . . . .	129
10.2	The normalization lemma . . . . .	130
10.3	Hilbert’s Nullstellensatz . . . . .	133

10.4	Jacobson rings . . . . .	136
10.5	Chains of prime ideals in geometric rings . . . . .	136
10.6	Height and dimension . . . . .	138
10.7	Dimension of geometric rings . . . . .	141
10.8	Exercises . . . . .	143
<b>11</b>	<b>Affine schemes</b>	<b>145</b>
11.1	The affine space $\mathbb{A}_n$ . . . . .	145
11.2	Affine schemes . . . . .	147
11.3	Closed and open subschemes of an affine scheme . . . . .	149
11.4	Functions defined on an open set . . . . .	153
11.5	Dimension of an affine scheme . . . . .	155
11.6	Irreducible components of an affine scheme . . . . .	156
11.7	Exercises . . . . .	157
<b>12</b>	<b>Morphisms of affine schemes</b>	<b>159</b>
12.1	Morphisms of affine schemes . . . . .	159
12.2	Immersions of affine schemes . . . . .	161
12.3	Local description of a morphism . . . . .	164
12.4	Product of affine schemes . . . . .	165
12.5	Dimension, product and intersection . . . . .	167
12.6	Dimension and fibres . . . . .	169
12.7	Finite morphisms . . . . .	170
12.8	Exercises . . . . .	171
<b>13</b>	<b>Zariski's main theorem</b>	<b>173</b>
13.1	Proof of Zariski's main theorem . . . . .	174
13.2	A factorization theorem . . . . .	178
13.3	Chevalley's semi-continuity theorem . . . . .	178
13.4	Exercises . . . . .	181
<b>14</b>	<b>Integrally closed Noetherian rings</b>	<b>183</b>
14.1	Reduced Noetherian rings . . . . .	183
14.2	Integrally closed Noetherian rings . . . . .	185
14.3	Discrete valuation rings. Dedekind rings . . . . .	187
14.4	Integral extensions of Noetherian domains . . . . .	190
14.5	Galois group and prime ideals . . . . .	192
14.6	Exercises . . . . .	194
<b>15</b>	<b>Weil divisors</b>	<b>197</b>
15.1	Weil divisors . . . . .	197
15.2	Reflexive rank-one modules and Weil divisors . . . . .	201
15.3	Exercises . . . . .	209

<b>16 Cartier divisors</b>	<b>211</b>
16.1 Cartier divisors . . . . .	211
16.2 The Picard group . . . . .	217
16.3 Exercises . . . . .	220
<i>Subject index</i>	<i>225</i>
<i>Symbols index</i>	<i>229</i>

# Introduction

Commutative algebra is the theory of commutative rings and their modules. Although it is an interesting theory in itself, it is generally seen as a tool for geometry and number theory. This is my point of view. In this book I try to organize and present a cohesive set of methods in commutative algebra, for use in geometry. As indicated in the title, I maintain throughout the text a view towards complex projective geometry.

In many recent algebraic geometry books, commutative algebra is often treated as a poor relation. One occasionally refers to it, but only reluctantly. It also suffers from having attracted too much attention thirty years ago. One or several texts are usually recommended: the “Introduction to Commutative Algebra” by Atiyah and Macdonald is a classic for beginners and Matsumura’s “Commutative rings” is better adapted for more advanced students. Both these books are excellent and most readers think that there is no need for any other. Today’s students seldom consult Bourbaki’s books on commutative algebra or the algebra part in the E.G.A. of Grothendieck and Dieudonné.

With this book, I want to prepare systematically the ground for an algebraic introduction to complex projective geometry. It is intended to be read by undergraduate students who have had a course in linear and multilinear algebra and know a bit about groups. They may have heard about commutative rings before, but apart from  $\mathbb{Z}$  and polynomial rings in one variable with coefficients in  $\mathbb{R}$  or  $\mathbb{C}$ , they have essentially worked with fields. I had to develop quite a lot of language new to them, but I have been careful to articulate all chapters around at least one important theorem. Furthermore I have tried to stimulate readers, whenever their attention may be drifting away, by presenting an example, or by giving them an exercise to solve.

In the first eight chapters, the general theory of rings and modules is developed. I put as much emphasis on modules as on rings; in modern algebraic geometry, sheaves and bundles play as important a role as varieties. I had to decide on the amount of homological algebra that should be included and on the form it should take. This is difficult since the border between commutative and homological algebras is not well-defined. I made several conventional choices. For example, I did not elaborate immediately on the homological

nature of length. But quite early on, when studying dualizing modules on Artinian rings in chapter six, I used non-elementary homological methods. In chapter seven, I have been particularly careful on rings and modules of fractions, hoping to prepare readers for working with sheaves.

I wanted this book to be self-contained. Consequently, the basic Galois theory had to be included. I slipped it in at the end of this first part, in chapter nine, just after the study of integral ring extensions.

Now, our favourite ring is  $\mathbb{C}[X_1, \dots, X_n]$ , the polynomial ring in several variables with coefficients in the field of complex numbers. We can derive many rings from this one by natural algebraic procedures, even though our purposes are geometric. Quotient rings of  $\mathbb{C}[X_1, \dots, X_n]$ , in other words, finitely generated  $\mathbb{C}$ -algebras, and their fraction rings are the basic objects from chapter ten to chapter thirteen. Noether's normalisation lemma and Hilbert's Nullstellensatz, two splendid theorems of commutative algebra, concern these rings and are at the heart of algebraic geometry. With these results in view, I discuss the notion of dimension and move heartily towards geometry by introducing affine complex schemes and their morphisms. I can then present and prove two other important geometric results, a local version of Zariski's main theorem and Chevalley's semi-continuity theorem.

From chapter fourteen on, I have tried to provide a solid background for modern intersection theory by presenting a detailed study of Weil and Cartier divisors.

In order to keep this book short, I have had to make many painful choices. Several of the chapters that I have deleted from this text will appear in a second book, intended for graduate students, and devoted to homological algebra and complex projective geometry.

I have been careless with the historical background. But I have been careful in developing the material slowly, at least initially, though it does become progressively more difficult as the text proceeds. When necessary, examples and exercises are included within chapters. I allow myself to refer to some of those. All chapters are followed by a series of exercises. Many are easy and a few are more intricate; readers will have to make their own evaluation!

I would like to thank the many students, undergraduates or graduates, whom I taught, or with whom I discussed algebra, at the universities of Strasbourg, Oslo and Paris VI. I have tried to attract each of them to algebra and algebraic geometry. Special thanks are due to Benedicte Basili who wrote a first set of notes from my graduate course in algebraic geometry and introduced me to LaTeX.

My wife Vivi has been tremendously patient while I was writing this first book. I hope very much that some readers will think that she did well in being so.



## Rings, homomorphisms, ideals

Our reader does not have to be familiar with commutative rings but should know their definition. Our rings always have an identity element 1. When necessary we write  $1_A$  for the identity element of  $A$ . The zero ring  $A = \{0\}$  is the only ring such that  $1_A = 0$ . In the first two sections we recall the really basic facts about ideals and homomorphisms (one of the reasons for doing so is because we need to agree on notation). From section 3 on, we begin to think about algebraic geometry. Prime and maximal ideals are the heart of the matter. Zariski topology, the radicals and comaximal ideals are henceforth treated. Our last section is a first approach to unique factorization domains (UFDs) (the proof of an essential theorem is postponed to chapter 7).

### Examples 1.1

1.  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$  and  $\mathbb{C}$  are rings. Each of them is a subring of the next.
2. A commutative field  $K$ , with identity element, is a non-zero ring such that  $K \setminus \{0\}$  is a multiplicative group.
3. The polynomial ring  $K[X_1, \dots, X_n]$  is a ring of which  $K[X_1, \dots, X_{n-1}]$  is a subring.
4. If  $A$  is a ring then  $A[X_1, \dots, X_n]$  is a ring of which  $A[X_1, \dots, X_{n-1}]$  is a subring.
5. If  $A$  and  $B$  are two rings, the product  $A \times B$  has a natural ring structure

$$(a, b) + (a', b') = (a + a', b + b') \quad \text{and} \quad (a, b)(a', b') = (aa', bb').$$

### Exercises 1.2

1. If  $K$  and  $K'$  are two fields, verify that the ring  $K \times K'$  is not a field.
2. Let  $p$  be a prime number. Denote by  $\mathbb{Z}_{(p)}$  the subset of  $\mathbb{Q}$  consisting of all  $n/m$  such that  $m \notin p\mathbb{Z}$ . Verify that  $\mathbb{Z}_{(p)}$  is a subring of  $\mathbb{Q}$ .

3. Let  $x = (x_1, \dots, x_n) \in K^n$  be a point. Verify that the set of all  $P/Q$ , with  $P, Q \in K[X_1, \dots, X_n]$ , and  $Q(x_1, \dots, x_n) \neq 0$ , is a subring of the field  $K(X_1, \dots, X_n)$ .

**Definition 1.3** Let  $A$  and  $B$  be rings. A (ring) homomorphism  $f : A \rightarrow B$  is a set application such that for all  $x, y \in A$ .

$$f(1_A) = 1_B, \quad f(x + y) = f(x) + f(y) \quad \text{and} \quad f(xy) = f(x)f(y).$$

An  $A$ -algebra is a ring  $B$  with a ring homomorphism  $f : A \rightarrow B$ .

The composition of two composable homomorphisms is clearly a homomorphism.

## 1.1 Ideals. Quotient rings

**Proposition 1.4** The kernel  $\ker f = f^{-1}(0)$ , of a ring homomorphism  $f : A \rightarrow B$  is a subgroup of  $A$  such that

$$(a \in \ker f \quad \text{and} \quad x \in A) \quad \Rightarrow \quad ax \in \ker f.$$

This is obvious.

**Definition 1.5** A subgroup  $\mathcal{I}$  of a ring  $A$  is an ideal of  $A$  if

$$(a \in \mathcal{I} \quad \text{and} \quad x \in A) \quad \Rightarrow \quad ax \in \mathcal{I}.$$

If  $\mathcal{I} \neq A$ , we say that  $\mathcal{I}$  is a proper ideal.

### Examples 1.6

1. The kernel of a ring homomorphism  $f : A \rightarrow B$  is an ideal of  $A$ .
2. If  $n \in \mathbb{Z}$ , then  $n\mathbb{Z}$  is an ideal of  $\mathbb{Z}$ .
3. Let  $a \in A$ . The set  $aA$  of all multiples of  $a$  is an ideal of  $A$ .
4. More generally, let  $a_i$ , with  $i \in E$ , be elements in  $A$ . The set of all linear combinations, with coefficients in  $A$ , of the elements  $a_i$  is an ideal. We say that the elements  $a_i$ ,  $i \in E$ , form a system of generators of (or generate) this ideal, which we often denote by  $((a_i)_{i \in E})$ .

As an obvious but important remark we note that all ideals contain 0.

**Exercise 1.7** Show that a ring  $A$  is a field if and only if  $(0)$  is the unique proper ideal of  $A$ .

**Theorem 1.8** *Let  $\mathcal{I}$  be an ideal of  $A$  and  $A/\mathcal{I}$  the quotient group of equivalence classes for the relation  $a \sim b \iff a - b \in \mathcal{I}$ . Then  $A/\mathcal{I}$  has a ring structure such that the class map  $\text{cl} : A \rightarrow A/\mathcal{I}$  is a ring homomorphism (obviously surjective) with kernel  $\mathcal{I}$ .*

*Proof* It is obvious that  $\text{cl}(a + b)$  and  $\text{cl}(ab)$  only depend on  $\text{cl}(a)$  and  $\text{cl}(b)$ . Defining then

$$\text{cl}(a) + \text{cl}(b) = \text{cl}(a + b) \quad \text{and} \quad \text{cl}(a)\text{cl}(b) = \text{cl}(ab),$$

the theorem is proved.  $\square$

**Definition 1.9** *The ring  $A/\mathcal{I}$  is the quotient ring of  $A$  by the ideal  $\mathcal{I}$ .*

**Example 1.10** If  $n \in \mathbb{Z}$ , the quotient ring (with  $|n|$  elements)  $\mathbb{Z}/n\mathbb{Z}$  is well known.

**Exercise 1.11** Let  $K$  be a field. Show that the composition homomorphism

$$K[Y] \xrightarrow{i} K[X, Y] \xrightarrow{\text{cl}} K[X, Y]/XK[X, Y]$$

is an isomorphism ( $i$  is the natural inclusion and  $\text{cl}$  the class application).

Let  $B$  be a quotient of  $A[X_1, \dots, X_n]$ . The obvious composition homomorphism  $A \rightarrow B$  gives to  $B$  the structure of an  $A$ -algebra. We note that any element in  $B$  is a combination, with coefficients in  $A$ , of products of the elements  $\text{cl}(X_1), \dots, \text{cl}(X_n) \in B$ . These elements generate  $B$  as an  $A$ -algebra.

**Definition 1.12** *A quotient ring  $B$  of a polynomial ring  $A[X_1, \dots, X_n]$  over a ring  $A$  is called an  $A$ -algebra of finite type or a finitely generated  $A$ -algebra.*

*Putting  $x_i = \text{cl}(X_i) \in B$ , we denote  $B$  by  $A[x_1, \dots, x_n]$  and we say that  $x_1, \dots, x_n$  generate  $B$  as an  $A$ -algebra.*

Clearly a quotient of an  $A$ -algebra of finite type is an  $A$ -algebra of finite type.

**Theorem 1.13** *(The factorization theorem)*

*Let  $f : A \rightarrow B$  be a ring homomorphism. There exists a unique injective ring homomorphism  $g : A/\ker f \rightarrow B$  such that the following diagram is commutative:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \text{cl} & \nearrow g & \\ A/\ker f & & \end{array}$$

*Furthermore  $f$  is surjective if and only if  $g$  is an isomorphism.*

*Proof* One verifies first that  $f(a)$  only depends on  $\text{cl}(a) \in A/\ker f$ . If we put then  $g(\text{cl}(a)) = f(a)$ , it is clear that  $g$  is a well-defined injective homomorphism. The rest of the theorem follows easily.  $\square$

The proof of the following proposition is straightforward and left to the reader.

**Proposition 1.14** *Let  $A$  be a ring and  $\mathcal{I}$  an ideal of  $A$ .*

- (i) *If  $\mathcal{J}$  is an ideal of  $A/\mathcal{I}$ , then  $\text{cl}^{-1}(\mathcal{J})$  is an ideal of  $A$  containing  $\mathcal{I}$ .*
- (ii) *If  $\mathcal{I}'$  is an ideal of  $A$  containing  $\mathcal{I}$ , then  $\text{cl}(\mathcal{I}')$  is an ideal of  $A/\mathcal{I}$  (denoted  $\mathcal{I}'/\mathcal{I}$ ).*
- (iii) *One has  $\text{cl}^{-1}(\text{cl}(\mathcal{I}')) = \mathcal{I}'$  and  $\text{cl}(\text{cl}^{-1}(\mathcal{J})) = \mathcal{J}$ . This bijection between the set of ideals of  $A$  containing  $\mathcal{I}$  and the set of ideals of  $A/\mathcal{I}$  respects inclusion.*

Note that this can be partly deduced from the next result which is useful by itself.

**Proposition 1.15** *If  $\mathcal{J}$  is an ideal of  $A/\mathcal{I}$ , then  $\text{cl}^{-1}(\mathcal{J})$  is the kernel of the composed ring homomorphism*

$$A \rightarrow A/\mathcal{I} \rightarrow (A/\mathcal{I})/\mathcal{J}.$$

*This homomorphism factorizes through an isomorphism*

$$A/\text{cl}^{-1}(\mathcal{J}) \simeq (A/\mathcal{I})/\mathcal{J}.$$

This description of  $\text{cl}^{-1}(\mathcal{J})$  needs no comment. The factorization is a consequence of the factorization theorem.

**Definition 1.16**

- (i) *An ideal generated by a finite number of elements is of finite type (or finitely generated).*
- (ii) *An ideal generated by one element is principal.*

If  $a_1, \dots, a_n$  generate the ideal  $\mathcal{J}$ , we write  $\mathcal{J} = (a_1, \dots, a_n)$ .

**Theorem 1.17**

- (i) *All ideals in  $\mathbb{Z}$  are principal.*

- (ii) If  $K$  is a field, all ideals in the polynomial ring (in one variable)  $K[X]$  are principal.

*Proof* Showing that a non-zero ideal  $\mathcal{I}$  of  $\mathbb{Z}$  is generated by the smallest positive integer of  $\mathcal{I}$  is straightforward.

Following the same principle, let  $\mathcal{I}$  be a non-zero ideal of  $K[X]$ . If  $P \in \mathcal{I}$  is a non-zero polynomial such that  $\deg(P) \leq \deg(Q)$  for all non-zero polynomials  $Q \in \mathcal{I}$ , showing that  $\mathcal{I} = PK[X]$  is also straightforward.  $\square$

**Definition 1.18** Let  $A$  be a ring.

- (i) If  $a \in A$  is invertible, in other words if there exists  $b \in A$  such that  $ab = 1$ , then  $a$  is a unit of  $A$ . One writes  $b = a^{-1}$  and says that  $b$  is the inverse of  $a$ .
- (ii) If  $a \in A$  and  $b \in B$  are elements such that  $ab = 0$  and  $b \neq 0$ , we say that  $a$  is a zero divisor.
- (iii) If  $a \in A$  is such that there exists an integer  $n > 0$  such that  $a^n = 0$ , then  $a$  is nilpotent.

**Examples 1.19**

1. The only units in  $\mathbb{Z}$  are 1 and  $-1$ .
2. The units of  $K[X]$  are the non-zero constants.
3. An element  $\text{cl}(m) \in \mathbb{Z}/n\mathbb{Z}$  is a unit if and only if  $m$  and  $n$  are relatively prime.
4. The ring  $\mathbb{Z}/n\mathbb{Z}$  has no zero divisors if and only if  $n$  is prime.
5. The ring  $\mathbb{Z}/n\mathbb{Z}$  has a non-zero nilpotent element if and only if  $n$  has a quadratic factor.
6. If  $\mathcal{I} = (X^2 + Y^2, XY) \subset K[X, Y]$ , then  $\text{cl}(X + Y)$ ,  $\text{cl}(X)$  and  $\text{cl}(Y)$  are nilpotent elements of  $K[X, Y]/\mathcal{I}$ .

**Definition 1.20**

- (i) A non-zero ring without zero divisors is called a domain.
- (ii) A non-zero ring without non-zero nilpotent elements is called a reduced ring.

**Definition 1.21** A domain which is not a field and such that all its ideals are principal is a principal ideal ring.

Hence our Theorem 1.17 can be stated in the following way.

**Theorem 1.22** The domains  $\mathbb{Z}$  and  $K[X]$  are principal ideal rings.

## 1.2 Operations on ideals

**Exercise 1.23** If  $\mathcal{I}$  and  $\mathcal{J}$  are ideals of a ring  $A$ , then  $\mathcal{I} \cap \mathcal{J}$  is an ideal of  $A$ .

Note that if  $\mathcal{I}$  and  $\mathcal{J}$  are ideals of a ring  $A$ , then  $\mathcal{I} \cup \mathcal{J}$  is not always an ideal of  $A$ .

**Definition 1.24** If  $\mathcal{I}$  and  $\mathcal{J}$  are ideals of a ring  $A$ , then

$$\mathcal{I} + \mathcal{J} = \{a + b, \quad a \in \mathcal{I}, b \in \mathcal{J}\}.$$

Note that  $\mathcal{I} + \mathcal{J}$  is an ideal of  $A$ , obviously the smallest ideal containing  $\mathcal{I}$  and  $\mathcal{J}$ .

**Definition 1.25** Let  $\mathcal{I}_s$  be a family of ideals of  $A$ . We denote by  $\sum_s \mathcal{I}_s$  the set formed by all finite sums  $\sum_s a_s$ , with  $a_s \in \mathcal{I}_s$ .

We note once more that  $\sum_s \mathcal{I}_s$  is an ideal of  $A$ , the smallest ideal containing  $\mathcal{I}_s$  for all  $s$ .

**Definition 1.26** If  $\mathcal{I}$  and  $\mathcal{J}$  are ideals of  $A$ , the product  $\mathcal{I}\mathcal{J}$  denotes the ideal generated by all  $ab$  with  $a \in \mathcal{I}$  and  $b \in \mathcal{J}$ .

**Definition 1.27** If  $\mathcal{I}$  is an ideal of  $A$  and  $P$  a subset of  $A$ , we denote by  $\mathcal{I} : P$  the set of all  $a \in A$  such that  $ax \in \mathcal{I}$  for all  $x \in P$ .

If  $P \not\subset \mathcal{I}$ , then  $\mathcal{I} : P$  is a proper ideal of  $A$ . If  $P \subset \mathcal{I}$ , then  $\mathcal{I} : P = A$ .

**Exercise 1.28** If  $\mathcal{I}_i$ ,  $i = 1, \dots, n$ , are ideals of  $A$  and  $P$  a subset of  $A$ , show that

$$\left(\bigcap_i \mathcal{I}_i\right) : P = \bigcap_i (\mathcal{I}_i : P).$$

**Proposition 1.29** If  $f : A \rightarrow B$  is a ring homomorphism and  $\mathcal{J}$  an ideal of  $B$ , then  $f^{-1}(\mathcal{J})$  is an ideal of  $A$  (the contraction of  $\mathcal{J}$  by  $f$ ).

*Proof* The kernel of the composition homomorphism  $A \rightarrow B \rightarrow B/\mathcal{J}$  is  $f^{-1}(\mathcal{J})$ .  $\square$

**Definition 1.30** If  $f : A \rightarrow B$  is a ring homomorphism and if  $\mathcal{I}$  is an ideal of  $A$ , we denote by  $f(\mathcal{I})B$  the ideal of  $B$  generated by the elements of  $f(\mathcal{I})$ .

In other words,  $f(\mathcal{I})B$  is the set consisting of all sums of elements of the form  $f(a)b$  with  $a \in \mathcal{I}$  and  $b \in B$ .

### 1.3 Prime ideals and maximal ideals

**Definition 1.31** *An ideal  $\mathcal{I}$  of  $A$  is prime if the quotient ring  $A/\mathcal{I}$  is a domain.*

We note that a prime ideal has to be proper. The following result is obvious.

**Proposition 1.32** *An ideal  $\mathcal{I}$  is prime if and only if*

$$ab \in \mathcal{I} \quad \text{and} \quad a \notin \mathcal{I} \quad \implies \quad b \in \mathcal{I}.$$

**Definition 1.33** *An ideal  $\mathcal{I}$  of  $A$  is maximal if the quotient ring  $A/\mathcal{I}$  is a field.*

Clearly, a maximal ideal is a prime ideal. The terminology is a bit unpleasant: obviously  $A$  is an absolute maximum in the set of all ideals, ordered by the inclusion. But the word maximal is justified by the following result:

**Proposition 1.34** *An ideal  $\mathcal{I}$  is maximal if and only if  $\mathcal{I}$  is a maximal element of the set of all proper ideals, ordered by the inclusion.*

*Proof* A field is a ring whose only ideal is  $(0)$ . Our proposition is an immediate consequence of Proposition 1.14.  $\square$

#### Exercises 1.35

1. Let  $k$  be a field and  $(a_1, \dots, a_n) \in k^n$ . Show that the set of all polynomials  $P \in k[X_1, \dots, X_n]$ , such that  $P(a_1, \dots, a_n) = 0$ , is a maximal ideal of  $k[X_1, \dots, X_n]$  generated by  $X_1 - a_1, \dots, X_n - a_n$ .
2. Show that all non-zero prime ideals of a principal ideal ring are maximal.

**Proposition 1.36** *Let  $\mathcal{P}$  be a prime ideal. If  $\mathcal{I}_i$ , with  $i = 1, \dots, n$ , are ideals such that  $\bigcap_1^n \mathcal{I}_i \subset \mathcal{P}$ , there exists  $l$  such that  $\mathcal{I}_l \subset \mathcal{P}$ .*

*Proof* Assume not; then there exist  $a_i \in \mathcal{I}_i$  and  $a_i \notin \mathcal{P}$  for  $i = 1, \dots, n$ . Since  $\mathcal{P}$  is a prime ideal, this implies  $\prod_1^n a_i \notin \mathcal{P}$ . But  $\prod_1^n a_i \in \bigcap_1^n \mathcal{I}_i$ ; this is a contradiction.  $\square$

**Theorem 1.37** (*Avoiding lemma*)

*Let  $\mathcal{I}_1, \dots, \mathcal{I}_n$  be ideals of  $A$  such that at most two are not prime. If  $\mathcal{J}$  is an ideal of  $A$  such that  $\mathcal{J} \not\subset \mathcal{I}_m$  for  $m = 1, \dots, n$ , then  $\mathcal{J} \not\subset \bigcup_1^n \mathcal{I}_m$ .*

*Proof* We use induction on  $n$ . The result is clear for  $n = 1$ .

If  $n > 1$ , by the induction hypothesis, there exists, for all  $i$ , an element  $a_i \in \mathcal{J}$ , such that  $a_i \notin \bigcup_{m \neq i} \mathcal{I}_m$ . We can obviously assume  $a_i \in \mathcal{I}_i$  for all  $i$ . Assume that  $\mathcal{I}_1$  is prime if  $n > 2$  and put  $a = a_1 + \prod_{i>1} a_i$ .

For  $i > 1$  we have  $a_1 \notin \mathcal{I}_i$  and  $\prod_{i>1} a_i \in \mathcal{I}_i$ . This shows  $a \notin \mathcal{I}_i$  for  $i > 1$ . Since  $\prod_{i>1} a_i \notin \mathcal{I}_1$  and  $a_1 \in \mathcal{I}_1$ , we have  $a \notin \mathcal{I}_1$  and we are done.  $\square$

Note that we have proved in fact the following more general useful result.

**Theorem 1.38** *Let  $\mathcal{I}_1, \dots, \mathcal{I}_n$  be ideals of  $A$  such that at most two are not prime. If  $E$  is a subset of  $A$ , stable for addition and multiplication, such that  $E \not\subset \mathcal{I}_m$  for  $m = 1, \dots, n$ , then  $E \not\subset \bigcup_1^n \mathcal{I}_m$ .*

**Proposition 1.39** *Let  $\mathcal{I}$  be an ideal of  $A$  and let  $\mathcal{J}$  be an ideal of  $A$  containing  $\mathcal{I}$ . Then  $\mathcal{J}$  is a prime (resp. maximal) ideal of  $A$  if and only if  $\mathcal{J}/\mathcal{I}$  is a prime (resp. maximal) ideal of  $A/\mathcal{I}$ .*

*Proof* The proposition is an immediate consequence of the ring isomorphism

$$A/\mathcal{J} \simeq (A/\mathcal{I})/(\mathcal{J}/\mathcal{I}).$$

$\square$

**Theorem 1.40** *A ring  $A \neq 0$  has a maximal ideal.*

Let us first recall Zorn's lemma (or axiom):

*Let  $E$  be a non-empty ordered set. If all totally ordered subsets of  $E$  are bounded above,  $E$  contains a maximal element.*

*Proof* Consider  $\mathcal{I}_i$ , a totally ordered set of proper ideals of  $A$ . Define  $\mathcal{I} = \bigcup_i \mathcal{I}_i$ . We show that  $\mathcal{I}$  is a proper ideal of  $A$  (obviously an upper bound for our totally ordered set).

If  $a, b \in \mathcal{I}$  and  $c \in A$ , there exists  $i \in E$  such that  $a, b \in \mathcal{I}_i$ . This implies  $a + b \in \mathcal{I}_i \subset \mathcal{I}$  and  $ac \in \mathcal{I}_i \subset \mathcal{I}$ . Furthermore, since  $1_A \notin \mathcal{I}_i$  for all  $i$ , it is clear that  $1_A \notin \mathcal{I}$ .  $\square$

Using Proposition 1.39, we get the following two corollaries.

**Corollary 1.41** *Any proper ideal of a ring is contained in a maximal ideal.*

**Corollary 1.42** *An element of a ring is invertible if and only if it is not contained in any maximal ideal of the ring.*



**Definition 1.43**

- (i) A ring with only one maximal ideal is local.
- (ii) If  $A$  and  $B$  are local rings with respective maximal ideals  $\mathcal{M}_A$  and  $\mathcal{M}_B$ , a homomorphism  $f : A \rightarrow B$  such that  $f(\mathcal{M}_A) \subset \mathcal{M}_B$  is called a local homomorphism of local rings.

**Exercises 1.44**

1. Show that the ring  $\mathbb{Z}_{(p)}$  (defined in Exercises 1.2) is local and that its maximal ideal is the set of all  $n/m$  with  $n \in p\mathbb{Z}$  and  $m \notin p\mathbb{Z}$ .
2. Let  $K$  be a field and  $(x_1, \dots, x_n) \in K^n$ . Show that the ring formed by all rational functions  $P/Q$ , with  $P, Q \in K[X_1, \dots, X_n]$  and such that  $Q(x_1, \dots, x_n) \neq 0$  is a local ring and that its maximal ideal is the set of all  $P/Q$  with  $P(x_1, \dots, x_n) = 0$  and  $Q(x_1, \dots, x_n) \neq 0$ .

**Definition 1.45** The spectrum  $\text{Spec}(A)$  of a ring  $A$  is the set of all prime ideals of  $A$ .

**Proposition 1.46** (Zariski topology)

If, for each ideal  $\mathcal{I}$  of  $A$ , we denote by  $V(\mathcal{I}) \subset \text{Spec}(A)$  the set of all prime ideals  $\mathcal{P}$  such that  $\mathcal{I} \subset \mathcal{P}$ , the subsets  $V(\mathcal{I})$  of  $\text{Spec}(A)$  are the closed sets of a topology on  $\text{Spec}(A)$ .

*Proof* If  $\mathcal{I}_s$ , with  $s = 1, \dots, n$ , are ideals of  $A$ , then  $\bigcup_1^n V(\mathcal{I}_s) = V(\bigcap_1^n \mathcal{I}_s)$ .

If  $\mathcal{I}_s$  is a family of ideals of  $A$ , then  $\bigcap_s V(\mathcal{I}_s) = V(\sum_s \mathcal{I}_s)$ . □

Note that by Corollary 1.41, a non-empty closed set of  $\text{Spec}(A)$  contains a maximal ideal.

**Definition 1.47** If  $s \in A$ , we denote by  $D(s)$  the open set  $\text{Spec}(A) \setminus V(sA)$  of  $\text{Spec}(A)$ .

We recall that a closed set of a topological space is irreducible if it is not the union of two strictly smaller closed subsets.

**Proposition 1.48** If  $\mathcal{P}$  is a prime ideal of a ring  $A$ , the closed set  $V(\mathcal{P}) \subset \text{Spec}(A)$  is irreducible.

*Proof* Let  $F_1$  and  $F_2$  be closed sets of  $\text{Spec}(A)$  such that  $V(\mathcal{P}) = F_1 \cup F_2$ . Then there exists  $i$  such that  $\mathcal{P} \in F_i$ . Consequently, we have  $V(\mathcal{P}) = F_i$ . □

As a special case, we get the following result.

**Proposition 1.49** Let  $A$  be a domain.

- (i) The topological space  $\text{Spec}(A)$  is irreducible.  
(ii) Any non-empty open subset of  $\text{Spec}(A)$  is dense in  $\text{Spec}(A)$ .

As we saw, the proof of this statement is straightforward. Its main consequence, which we will understand in due time, is that algebraic varieties are irreducible topological spaces for the Zariski topology.

## 1.4 Nilradicals and Jacobson radicals

**Proposition 1.50** *The nilpotent elements of a ring  $A$  form an ideal of  $A$ .*

*Proof* If  $a^n = 0$  and  $b^m = 0$ , then  $(ca - db)^{n+m-1} = 0$ . □

**Corollary 1.51** *If  $\mathcal{I}$  is an ideal of  $A$ , the set of all elements  $a \in A$  having a power in  $\mathcal{I}$  is an ideal of  $A$ .*

*Proof* Apply the proposition to the ring  $A/\mathcal{I}$ . □

**Definition 1.52** *This ideal is the radical  $\sqrt{\mathcal{I}}$  of  $\mathcal{I}$ . The radical  $\sqrt{(0)}$  of  $(0)$  is the nilradical  $\text{Nil}(A)$  of  $A$ .*

Note that  $\sqrt{\sqrt{\mathcal{I}}} = \sqrt{\mathcal{I}}$ . As a consequence we see that if  $A$  is not the zero ring, then  $A/\sqrt{(0)}$  is reduced.

**Proposition 1.53** *A non-zero ring  $A$  is reduced if and only if  $\text{Nil}(A) = (0)$ .*

This is the definition of a reduced ring.

**Theorem 1.54** *If  $A$  is not the zero ring, the nilradical  $\text{Nil}(A)$  is the intersection of all prime ideals of  $A$ .*

*Proof* Consider  $a \in \text{Nil}(A)$  and  $\mathcal{P}$  a prime ideal. There exists  $n > 0$  such that  $a^n = 0$ , hence  $a^n \in \mathcal{P}$  and  $a \in \mathcal{P}$ . This proves  $\text{Nil}(A) \subset \mathcal{P}$ .

Assume now  $a \notin \text{Nil}(A)$  and let us show that there exists a prime ideal  $\mathcal{P}$  such that  $a \notin \mathcal{P}$ . Consider the part  $S$  of  $A$  consisting of all positive powers of  $a$ . We have assumed that  $0 \notin S$ . We can therefore consider the non-empty set  $E$  of all ideals of  $A$  which do not intersect  $S$ . Let  $E'$  be a totally ordered subset of  $E$ . Clearly  $E'$  is bounded above, in  $E$ , by  $\bigcup_{\mathcal{I} \in E'} \mathcal{I}$ . By Zorn's lemma,  $E$  has a maximal element.

If  $\mathcal{I}$  is a maximal element in  $E$ , let us show that  $\mathcal{I}$  is a prime ideal.

Let  $x, y \in A$  be such that  $xy \in \mathcal{I}$ . If  $x \notin \mathcal{I}$  and  $y \notin \mathcal{I}$ , there are positive integers  $n$  and  $m$  such that  $a^n \in \mathcal{I} + xA$  and  $a^m \in \mathcal{I} + yA$ . This implies

$$a^{n+m} \in \mathcal{I} + x\mathcal{I} + y\mathcal{I} + xyA \subset \mathcal{I},$$

hence a contradiction. □