Cambridge studies in advanced mathematics 113

An Introduction to Lie Groups and Lie Algebras

ALEXANDER KIRILLOV, JR.

CAMBRIDGE www.cambridge.org/9780521889698

This page intentionally left blank

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS 113

An Introduction to Lie Groups and Lie Algebras

With roots in the nineteenth century, Lie theory has since found many and varied applications in mathematics and mathematical physics, to the point where it is now regarded as a classical branch of mathematics in its own right. This graduate text focuses on the study of semisimple Lie algebras, developing the necessary theory along the way.

The material covered ranges from basic definitions of Lie groups, to the theory of root systems, and classification of finite-dimensional representations of semisimple Lie algebras. Written in an informal style, this is a contemporary introduction to the subject which emphasizes the main concepts of the proofs and outlines the necessary technical details, allowing the material to be conveyed concisely.

Based on a lecture course given by the author at the State University of New York at Stony Brook, the book includes numerous exercises and worked examples and is ideal for graduate courses on Lie groups and Lie algebras.

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit: http://www.cambridge.org/series/sSeries.asp?code=CSAM

Already published

60 M. P. Brodmann & R. Y. Sharp *Local cohomology*
61 J. D. Dixon *et al. Analytic pro-p groups*
62 R. Stanley *Enumerative combinatorics II*
63 R. M. Dudley *Uniform central limit theorems*
64 J. Jost & X. Li-Jost *Calculus of variations*
65 A. J. Berrick & M. E. Keating *An introduction to rings and modules*
66 S. Morosawa *Holomorphic dynamics*
67 A. J. Berrick & M. E. Keating *Categories and modules with K-theory in view*
68 K. Sato *Levy processes and infinitely divisible distributions*
69 H. Hida *Modular forms and Galois cohomology*
70 R. Iorio & V. Iorio *Fourier analysis and partial differential equations*
71 R. Blei *Analysis in integer and fractional dimensions*
72 F. Borceaux & G. Janelidze *Galois theories*
73 B. Bollobás *Random graphs*
74 R. M. Dudley *Real analysis and probability*
75 T. Sheil-Small *Complex polynomials*
76 C. Voisin *Hodge theory and complex algebraic geometry, I*
77 C. Voisin *Hodge theory and complex algebraic geometry, II*
78 V. Paulsen *Completely bounded maps and operator algebras*
79 F. Gesztesy & H. Holden *Soliton Equations and Their Algebro-Geometric Solutions, I*
81 S. Mukai *An Introduction to Invariants and Moduli*
82 G. Tourlakis *Lectures in Logic and Set Theory, I*
83 G. Tourlakis *Lectures in Logic and Set Theory, II*
84 R. A. Bailey *Association Schemes*
85 J. Carison, S. Müller-Stach & C. Peters *Period Mappings and Period Domains*
86 J. J. Duistermaat & J. A. C. Kolk *Multidimensional Real Analysis I*
87 J. J. Duistermaat & J. A. C. Kolk *Multidimensional Real Analysis II*
89 M. Golumbic & A. Trenk *Tolerance Graphs*
90 L. Harper *Global Methods for Combinatorial Isoperimetric Problems*
91 I. Moerdijk & J. Mrcun *Introduction to Foliations and Lie Groupoids*
92 J. Kollar, K. E. Smith & A. Corti *Rational and Nearly Rational Varieties*
93 D. Applebaum *Levy Processes and Stochastic Calculus*
94 B. Conrad *Modular Forms and the Ramanujan Conjecture*
95 M. Schechter *An Introduction to Nonlinear Analysis*
96 R. Carter *Lie Algebras of Finite and Affine Type*
97 H. L. Montgomery, R. C. Vaughan & M. Schechter *Multiplicative Number Theory I*
98 I. Chavel *Riemannian Geometry*
99 D. Goldfeld *Automorphic Forms and L-Functions for the Group GL(n,R)*
100 M. Marcus & J. Rosen *Markov Processes. Gaussian Processes, and Local Times*
101 P. Gille & T. Szamuely *Central Simple Algebras and Galois Cohomology*
102 J. Bertoin *Random Fragmentation and Coagulation Processes*
103 E. Frenkel *Langlands Correspondence for Loop Groups*
104 A. Ambrosetti & A. Malchiodi *Nonlinear Analysis and Semilinear Elliptic Problems*
105 T. Tao & V. H. Vu *Additive Combinatorics*
106 E. B. Davies *Linear Operators and their Spectra*
107 K. Kodaira *Complex Analysis*
108 T. Ceccherini-Silberstein, F. Scarabotti & F. Tolli *Harmonic Analysis on Finite Groups*
109 H. Geiges *An Introduction to Contact Topology*
110 J. Faraut *Analysis on Lie Groups*
111 E. Park *Complex Topological K-Theory*
112 D. W. Stroock *Partial Differential Equations for Probabilists*
113 A. Kirillov *An Introduction to Lie Groups and Lie Algebras*

An Introduction to Lie Groups and Lie Algebras

ALEXANDER KIRILLOV, Jr.

Department of Mathematics, SUNY at Stony Brook

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK
Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521889698

First published in print format 2008

ISBN-13 978-0-511-42319-2 eBook (EBL)

ISBN-13 978-0-521-88969-8 hardback

Dedicated to my teachers

Contents

Preface

This book is an introduction to the theory of Lie groups and Lie algebras, with emphasis on the theory of semisimple Lie algebras. It can serve as a basis for a two-semester graduate course or – omitting some material – as a basis for a rather intensive one-semester course. The book includes a large number of exercises.

The material covered in the book ranges from basic definitions of Lie groups to the theory of root systems and highest weight representations of semisimple Lie algebras; however, to keep book size small, the structure theory of semisimple and compact Lie groups is not covered.

Exposition follows the style of famous Serre's textbook on Lie algebras [47]: we tried to make the book more readable by stressing ideas of the proofs rather than technical details. In many cases, details of the proofs are given in exercises (always providing sufficient hints so that good students should have no difficulty completing the proof). In some cases, technical proofs are omitted altogether; for example, we do not give proofs of Engel's or Poincare–Birkhoff–Witt theorems, instead providing an outline of the proof. Of course, in such cases we give references to books containing full proofs.

It is assumed that the reader is familiar with basics of topology and differential geometry (manifolds, vector fields, differential forms, fundamental groups, covering spaces) and basic algebra (rings, modules). Some parts of the book require knowledge of basic homological algebra (short and long exact sequences, Ext spaces).

Errata for this book are available on the book web page at
http://www.math.sunysb.edu/~kirillov/liegroups/.

1
Introduction

In any algebra textbook, the study of group theory is usually mostly concerned with the theory of finite, or at least finitely generated, groups. This is understandable: such groups are much easier to describe. However, most groups which appear as groups of symmetries of various geometric objects are not finite: for example, the group $SO(3, \mathbb{R})$ of all rotations of three-dimensional space is not finite and is not even finitely generated. Thus, much of material learned in basic algebra course does not apply here; for example, it is not clear whether, say, the set of all morphisms between such groups can be explicitly described.

The theory of Lie groups answers these questions by replacing the notion of a finitely generated group by that of a Lie group – a group which at the same time is a finite-dimensional manifold. It turns out that in many ways such groups can be described and studied as easily as finitely generated groups – or even easier. The key role is played by the notion of a Lie algebra, the tangent space to G at identity. It turns out that the group operation on G defines a certain bilinear skew-symmetric operation on $\mathfrak{g} = T_1 G$; axiomatizing the properties of this operation gives a definition of a Lie algebra.

The fundamental result of the theory of Lie groups is that many properties of Lie groups are completely determined by the properties of corresponding Lie algebras. For example, the set of morphisms between two (connected and simply connected) Lie groups is the same as the set of morphisms between the corresponding Lie algebras; thus, describing them is essentially reduced to a linear algebra problem.

Similarly, Lie algebras also provide a key to the study of the structure of Lie groups and their representations. In particular, this allows one to get a complete classification of a large class of Lie groups (semisimple and more generally, reductive Lie groups; this includes all compact Lie groups and all classical Lie groups such as $SO(n, \mathbb{R})$) in terms of relatively simple geometric objects, so-called root systems. This result is considered by many mathematicians (including the author of this book) to be one of the most beautiful achievements in all of mathematics. We will cover it in Chapter 7.

To conclude this introduction, we will give a simple example which shows how Lie groups naturally appear as groups of symmetries of various objects – and how one can use the theory of Lie groups and Lie algebras to make use of these symmetries.

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. Define the Laplace operator $\Delta_{\text{sph}} : C^\infty(S^2) \to C^\infty(S^2)$ by $\Delta_{\text{sph}} f = (\Delta\tilde{f})|_{S^2}$, where $\tilde{f}$ is the result of extending f to $\mathbb{R}^3 - \{0\}$ (constant along each ray), and Δ is the usual Laplace operator in $\mathbb{R}^3$. It is easy to see that Δ_{sph} is a second-order differential operator on the sphere; one can write explicit formulas for it in the spherical coordinates, but they are not particularly nice.

For many applications, it is important to know the eigenvalues and eigenfunctions of Δ_{sph}. In particular, this problem arises in quantum mechanics: the eigenvalues are related to the energy levels of a hydrogen atom in quantum mechanical description. Unfortunately, trying to find the eigenfunctions by brute force gives a second-order differential equation which is very difficult to solve.

However, it is easy to notice that this problem has some symmetry – namely, the group $\mathrm{SO}(3, \mathbb{R})$ acting on the sphere by rotations. How can one use this symmetry?

If we had just one symmetry, given by some rotation $R: S^2 \to S^2$, we could consider its action on the space of complex-valued functions $C^\infty(S^2, \mathbb{C})$. If we could diagonalize this operator, this would help us study Δ_{sph}: it is a general result of linear algebra that if A, B are two commuting operators, and A is diagonalizable, then B must preserve eigenspaces for A. Applying this to pair R, Δ_{sph}, we get that Δ_{sph} preserves eigenspaces for R, so we can diagonalize Δ_{sph} independently in each of the eigenspaces.

However, this will not solve the problem: for each individual rotation R, the eigenspaces will still be too large (in fact, infinite-dimensional), so diagonalizing Δ_{sph} in each of them is not very easy either. This is not surprising: after all, we only used one of many symmetries. Can we use all of rotations $R \in \mathrm{SO}(3, \mathbb{R})$ simultaneously?

This, however, presents two problems.

- $\mathrm{SO}(3, \mathbb{R})$ is not a finitely generated group, so apparently we will need to use infinitely (in fact uncountably) many different symmetries and diagonalize each of them.
- $\mathrm{SO}(3, \mathbb{R})$ is not commutative, so different operators from $\mathrm{SO}(3, \mathbb{R})$ can not be diagonalized simultaneously.

The goal of the theory of Lie groups is to give tools to deal with these (and similar) problems. In short, the answer to the first problem is that $\mathrm{SO}(3,\mathbb{R})$ is in a certain sense finitely generated – namely, it is generated by three generators, “infinitesimal rotations” around x, y, z axes (see details in Example 3.10).

The answer to the second problem is that instead of decomposing the $C^\infty(S^2,\mathbb{C})$ into a direct sum of common eigenspaces for operators $R \in \mathrm{SO}(3,\mathbb{R})$, we need to decompose it into “irreducible representations” of $\mathrm{SO}(3,\mathbb{R})$. In order to do this, we need to develop the theory of representations of $\mathrm{SO}(3,\mathbb{R})$. We will do this and complete the analysis of this example in Section 4.8.

2

Lie groups: basic definitions

2.1. Reminders from differential geometry

This book assumes that the reader is familiar with basic notions of differential geometry, as covered for example, in [49]. For reader's convenience, in this section we briefly remind some definitions and fix notation.

Unless otherwise specified, all manifolds considered in this book will be C^∞ real manifolds; the word "smooth" will mean C^∞. All manifolds we will consider will have at most countably many connected components.

For a manifold M and a point $m \in M$, we denote by $T_m M$ the tangent space to M at point m, and by TM the tangent bundle to M. The space of vector fields on M (i.e., global sections of TM) is denoted by $\mathrm{Vect}(M)$. For a morphism $f: X \to Y$ and a point $x \in X$, we denote by $f_*: T_x X \to T_{f(x)} Y$ the corresponding map of tangent spaces.

Recall that a morphism $f: X \to Y$ is called an *immersion* if $\operatorname{rank} f_* = \dim X$ for every point $x \in X$; in this case, one can choose local coordinates in a neighborhood of $x \in X$ and in a neighborhood of $f(x) \in Y$ such that f is given by $f(x_1, \dots x_n) = (x_1, \dots, x_n, 0, \dots 0)$.

An *immersed submanifold* in a manifold M is a subset $N \subset M$ with a structure of a manifold (not necessarily the one inherited from M!) such that inclusion map $i: N \hookrightarrow M$ is an immersion. Note that the manifold structure on N is part of the data: in general, it is not unique. However, it is usually suppressed in the notation. Note also that for any point $p \in N$, the tangent space to N is naturally a subspace of tangent space to M: $T_p N \subset T_p M$.

An *embedded submanifold* $N \subset M$ is an immersed submanifold such that the inclusion map $i: N \hookrightarrow M$ is a homeomorphism. In this case the smooth structure on N is uniquely determined by the smooth structure on M.

Following Spivak, we will use the word "submanifold" for *embedded* submanifolds (note that many books use word submanifold for immersed submanifolds).

All of the notions above have complex analogs, in which real manifolds are replaced by complex analytic manifolds and smooth maps by holomorphic maps. We refer the reader to [49] for details.

2.2. Lie groups, subgroups, and cosets

Definition 2.1. A (real) Lie group is a set G with two structures: G is a group and G is a manifold. These structures agree in the following sense: multiplication map $G \times G \to G$ and inversion map $G \to G$ are smooth maps.

A morphism of Lie groups is a smooth map which also preserves the group operation: $f(gh) = f(g)f(h), f(1) = 1$. We will use the standard notation $\operatorname{Im} f$, $\operatorname{Ker} f$ for image and kernel of a morphism.

The word "real" is used to distinguish these Lie groups from complex Lie groups defined below. However, it is frequently omitted: unless one wants to stress the difference with complex case, it is common to refer to real Lie groups as simply Lie groups.

Remark 2.2. One can also consider other classes of manifolds: C^1, C^2, analytic. It turns out that all of them are equivalent: every C^0 Lie group has a unique analytic structure. This is a highly non-trivial result (it was one of Hilbert's 20 problems), and we are not going to prove it (the proof can be found in the book [39]). Proof of a weaker result, that C^2 implies analyticity, is much easier and can be found in [10, Section 1.6]. In this book, "smooth" will be always understood as C^∞.

In a similar way, one defines complex Lie groups.

Definition 2.3. A complex Lie group is a set G with two structures: G is a group and G is a complex analytic manifold. These structures agree in the following sense: multiplication map $G \times G \to G$ and inversion map $G \to G$ are analytic maps.

A morphism of complex Lie groups is an analytic map which also preserves the group operation: $f(gh) = f(g)f(h), f(1) = 1$.

Remark 2.4. Throughout this book, we try to treat both real and complex cases simultaneously. Thus, most theorems in this book apply both to real and complex Lie groups. In such cases, we will say "let G be real or complex Lie group …" or "let G be a Lie group over $\mathbb{K}$ …", where $\mathbb{K}$ is the base field: $\mathbb{K} = \mathbb{R}$ for real Lie groups and $\mathbb{K} = \mathbb{C}$ for complex Lie groups.

When talking about complex Lie groups, "submanifold" will mean "complex analytic submanifold", tangent spaces will be considered as complex vector spaces, all morphisms between manifolds will be assumed holomorphic, etc.

Example 2.5. The following are examples of Lie groups:

(1) $\mathbb{R}^n$, with the group operation given by addition
(2) $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, $\times$
$\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$, $\times$
(3) $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, $\times$
(4) $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$. Many of the groups we will consider will be subgroups of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$.
(5) $\mathrm{SU}(2) = \{A \in \mathrm{GL}(2, \mathbb{C}) \mid A\bar{A}^t = 1, \det A = 1\}$. Indeed, one can easily see that

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Writing $\alpha = x_1 + \mathrm{i}x_2, \beta = x_3 + \mathrm{i}x_4, x_i \in \mathbb{R}$, we see that SU(2) is diffeomorphic to $S^3 = \{x_1^2 + \cdots + x_4^2 = 1\} \subset \mathbb{R}^4$.
(6) In fact, all usual groups of linear algebra, such as $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{O}(n, \mathbb{R})$, $\mathrm{U}(n)$, $\mathrm{SO}(n, \mathbb{R})$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n, \mathbb{R})$ are (real or complex) Lie groups. This will be proved later (see Section 2.7).

Note that the definition of a Lie group does not require that G be connected. Thus, any finite group is a 0-dimensional Lie group. Since the theory of finite groups is complicated enough, it makes sense to separate the finite (or, more generally, discrete) part. It can be done as follows.

Theorem 2.6. *Let G be a real or complex Lie group. Denote by G^0 the connected component of identity. Then G^0 is a normal subgroup of G and is a Lie group itself* (*real or complex, respectively*). *The quotient group G/G^0 is discrete.*

Proof. We need to show that G^0 is closed under the operations of multiplication and inversion. Since the image of a connected topological space under a continuous map is connected, the inversion map i must take G^0 to one component of G, that which contains $i(1) = 1$, namely G^0. In a similar way one shows that G^0 is closed under multiplication.

To check that this is a normal subgroup, we must show that if $g \in G$ and $h \in G^0$, then $ghg^{-1} \in G^0$. Conjugation by g is continuous and thus will take G^0 to some connected component of G; since it fixes 1, this component is G^0.

The fact that the quotient is discrete is obvious. □