Application of Distributions to the Theory of Elementary Particles in Quantum Mechanics

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Editors' Preface

SEVENTY years ago when the fraternity of physicists was smaller than the audience at a weekly physics colloquium in a major university a J. Willard Gibbs could, after ten years of thought, summarize his ideas on a subject in a few monumental papers or in a classic treatise. His competition did not intimidate him into a muddled correspondence with his favorite editor, nor did it occur to his colleagues that their own progress was retarded by his leisurely publication schedule.

Today the dramatic phase of a new branch of physics spans less than a decade and subsides before the definitive treatise is published. Moreover modern physics is an extremely interconnected discipline and the busy practitioner of one of its branches must be kept aware of breakthroughs in other areas. An expository literature which is clear and timely is needed to relieve him of the burden of wading through tentative and hastily written papers scattered in many journals.

To this end we have undertaken the editing of a new series, entitled *Documents on Modern Physics*, which will make available selected reviews, lecture notes, conference proceedings, and important collections of papers in branches of physics of special current interest. Complete coverage of a field will not be a primary aim. Rather, we will emphasize readability, speed of publication, and importance to students and research workers. The books will appear in low-cost paper-covered editions, as well as in cloth covers. The scope will be broad, the style informal.

From time to time, older branches of physics come alive again, and forgotten writings acquire relevance to recent developments. We expect to make a number of such works available by including them in this series along with new works.

ELLIOTT MONTROLL
GEORGE H. VINEYARD
MAURICE LÉVY

Preface

I gave lectures in 1958 in the National University of Argentine, Buenos-Aires, and in 1960, in the University of California, Berkeley, on application of distributions to the study of relativistic elementary particles. These lectures were mimeographed by the Universities concerned; both texts are now out of print. The English text has been translated into Russian (MIR Publications, Moscow 1964). This book is a revised version of the Berkeley lectures notes of 1960. Therefore it is not written and planned as a textbook. Each part was written after the lecture had been given—hence some repetitions or even modifications in the book. This text contains a large amount of pure mathematics: theory of vector valued distributions, tensor products of topological vector spaces, Fourier transforms and Bochner theorem, Hilbert sub-spaces and associated kernels have been exposed as such, but many proofs have been omitted, being intended primarily for physicists. I tried to make a distinction between scalar valued functions or distributions printed in normal characters, and vector valued, printed in bold characters. But this tedious distinction has been progressively abandoned throughout, and everything is written in normal letters, even for vectors, except in special cases where the distinction is important.

L. SCHWARTZ

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CHAPTER 1

Position of the Problem

Introduction

Quantum mechanics deals with the description of motions of particles. All the information needed for the complete description of the motion of a particle is contained in its wave function $\psi(x, y, z, t)$, a complex function of position $(x, y, z) \in R^3$ (three-dimensional Euclidean space) and the time t. In non-relativistic quantum mechanics $|\psi(x, y, z, t)|^2$ is the probability density of the position of the particle. The probability of the position of a particle being in a region $A \subset R^3$ at any time t is $\iiint_A |\psi(x, y, z, t)|^2 \, dx\,dy\,dz$. Note ψ must be square integrable for each t and we assume $\iiint_{R^3} |\psi(x, y, z, t)|^2 \, dx\,dy\,dz = 1$ for each t. If we define an inner product $\iiint_{R^3} \psi_1(x, y, z, t)\overline{\psi_2(x, y, z, t)} \, dx\,dy\,dz$, then the function ψ belongs to a Hilbert space for each t.

In non-relativistic quantum mechanics, ψ satisfies the Schroedinger wave equation:

$$i\hbar\,(\partial\psi/\partial t) = H\psi$$

where $\hbar$ is Plank's constant divided by 2π, and H is a self-adjoint operator in the Hilbert space L^2 of square-integrable functions on R^3. It follows from the Schroedinger equation that the inner product of two wave functions remains constant for all time. When the particle is free of interaction,

$$H = -\frac{\hbar^2}{2m}\Delta$$

where m is the mass of the particle and Δ is the Laplacian.

In relativistic quantum mechanics, space and time are not separate; thus one cannot say that ψ is a function of four variables, unless a Lorentz coordinate system is chosen. In order to treat space and time together, the space E_4, a four-dimensional affine space, is introduced and ψ is defined on E_4. An affine space will be defined later.

Definition. A *particle*, $\mathscr{H}$, is a Hilbert space of functions on E_4.

Definition. A motion, ψ, is an element of $\mathscr{H}$ with $\|\psi\| = 1$.

Let σ be an arbitrary Lorentz transformation in E_4 and G be the Lorentz group. Under the transformation σ, a function ψ goes into a function $\sigma\psi$.

Definition. If, for all $\sigma \in G$,

$$\psi \in \mathscr{H} \Rightarrow \sigma\psi \in \mathscr{H}$$

$$\|\sigma \psi\|_{\mathscr{H}} = \|\psi\|_{\mathscr{H}}$$

then the particle, $\mathscr{H}$, is a *universal particle.* In short, a universal particle is a particle that does not change under a Lorentz transformation.

Definition. A universal particle, $\mathscr{H}$, is *elementary* if $\mathscr{H}$ contains no subspace which transforms into itself under all $\sigma \in G$, i.e., $\mathscr{H}$ is minimal.

We shall show later that the space $\mathscr{H}$ depends on a parameter $m_0 \geqslant 0$ and a parameter taking on the two values $+$ and $-$. The $\pm$ parameter is interpreted as the charge and m_0 as the rest mass of the particle.

Definition. A *meson* is a scalar elementary particle (i.e., the wave function ψ is a scalar).

For a system of two particles, the Hilbert space has the same axioms as before, except that its elements are functions on $E_4 \times E_4$. Only systems of one free particle will be dealt with in these lectures.

For the sake of generality, we shall assume that our Hilbert space is not a space of functions but a space of distributions.

We, therefore, begin with a short introduction to the theory of distributions.

Elements of the Theory of Distributions

Let R^n denote the n-dimensional Euclidean space and let $\mathscr{D}(R^n)$ (or simply $\mathscr{D}$) be the space of all complex valued functions φ defined in R^n which have derivatives of all orders and which vanish identically outside a bounded region in R^n. The functions φ will be called *testing functions.* Note that $\mathscr{D}(R^n)$ is a linear space.

We introduce now a topology in $\mathscr{D}$.

Definition. A sequence of testing functions $\{\varphi_j(x)\}$ *converges to zero in* $\mathscr{D}$ if all the functions $\varphi_j(x)$ vanish identically outside the same bounded region in R^n and if the functions $\varphi_j(x)$ and all their derivatives converge uniformly to zero.

Definition. A *distribution* T is a continuous linear functional on $\mathscr{D}$, i.e., the image under T of an element $\varphi \in \mathscr{D}$ is a complex number denoted by $\langle T, \phi \rangle$ such that

$$\langle T, (c_1\varphi_1 + c_2\varphi_2)\rangle = c_1\langle T, \varphi_1\rangle + c_2\langle T, \varphi_2\rangle$$

and

$$\varphi_j \underset{\mathscr{D}}{\rightarrow} 0 \text{ implies that } \langle T, \varphi_j\rangle \rightarrow 0.$$

Let $\mathscr{D}'(R^n)$ (or simply $\mathscr{D}'$) denote the space of distributions on R^n.

Example. Let f be a locally integrable function in R^n.
Then

$$\langle f, \varphi\rangle = \int_{R^n} f(x)\varphi(x)\,\mathrm{d}x = \int_A f(x)\varphi(x)\,\mathrm{d}x$$

defines a distribution. Here A is a bounded region in R^n (the support of φ). Thus every locally integrable function defines a distribution. Clearly, f_1 and f_2 define the same distribution if and only if $f_1 = f_2$ almost everywhere. Considering the Lebesgue classes defined by this relation (i.e., identifying functions which are equal almost everywhere), we conclude that the Lebesgue classes of locally integrable functions form a subspace of the space of distributions.

Other important examples are the Dirac distribution, δ, defined by

$$\langle \delta, \varphi\rangle = \varphi(0)$$

or

$$\langle \delta(a), \varphi\rangle = \varphi(a)$$

and the dipole ζ defined by

$$\langle \zeta, \varphi\rangle = -\varphi'(0).$$

Definition. The *derivative of a distribution* T is defined by the formula:

$$\langle T', \varphi\rangle = -\langle T, \varphi'\rangle$$

From this it follows that

$$\langle T^{(m)}, \varphi\rangle = (-1)^m \langle T, \varphi^{(m)}\rangle$$

$$\langle \frac{\partial T}{\partial x_k}, \varphi\rangle = -\langle T, \frac{\partial \varphi}{\partial x_k}\rangle$$

$$\langle D^p T, \varphi\rangle = (-1)^{|p|} \langle T, D^p\varphi\rangle$$

where p denotes the n-tuple of integers $p = (p_1, \ldots, p_n)$,

$$|p| = p_1 + \ldots + p_n$$

and

$$D^p = \left(\frac{\partial}{\partial x_1}\right)^{p_1} \left(\frac{\partial}{\partial x_2}\right)^{p_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{p_n}.$$

Thus every distribution has derivatives of all orders.

Example. Consider the Heaviside function $Y(x)$ defined by

$$Y(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

Then

$$\langle Y', \varphi \rangle = -\langle Y, \varphi' \rangle = -\int_{-\infty}^{\infty} Y(x)\, \varphi'(x)\, dx$$

$$= -\int_{0}^{\infty} \varphi'(x)\, dx = \varphi(0) = \langle \delta, \varphi \rangle$$

Therefore $Y' = \delta$.

Definition. Let f be a continuous function and let $A = \{x : f(x) \neq 0\}$. The closure $\overline{A}$ of A is called the *support* of the function f.

Definition. Let Ω be an open set in R^n and let $T \in \mathscr{D}'$. We say that $T = 0$ in Ω if $\langle T, \varphi \rangle = 0$ for all $\varphi \in \mathscr{D}$ whose support is contained in Ω. For example, $\delta = 0$ in $R - \{0\}$.

Theorem. Let $\{\Omega_i\}$ be any system of open subsets in R^n and suppose that $T = 0$ in every Ω_i. Then $T = 0$ in $\cup \Omega_i$.

Proof. We must show that $\langle T, \varphi \rangle = 0$ for every $\varphi \in \mathscr{D}$ whose support is contained in $\cup \Omega_i$. Let A be the support of some $\varphi \in \mathscr{D}$. Since A is compact and covered by $\{\Omega_i\}$, there exists a finite subcover $\{\Omega_{i_k}\}$, $k = 1, \ldots, n$. Let $\{\psi_k\}$, $k = 1, \ldots, n$, be an infinitely many times continuously differentiable partition of unity on A with respect to Ω_{i_k}, that is, $\psi_k \in \mathscr{D}(R^n)$, each ψ_k has its support in Ω_{i_k} and

$$\sum_{k=1}^{n} \psi_k = 1$$

on A. Then

$$\langle T, \varphi \rangle = \langle T, \sum_{k=1}^{n} \psi_k \varphi \rangle = \sum_{k=1}^{n} \langle T, \psi_k \varphi \rangle = 0.$$

Corollary. For every distribution T there exists exactly one maximal open subset of R^n in which T is zero.

Proof. Consider all Ω_i in which $T = 0$. Then $\cup\, \Omega_i$ is the required set.

Definition. The *support* of T is the complement of the maximal open subset of R^n in which $T = 0$.

We introduce now a topology in the space of distribution $\mathscr{D}'$. Since it is a linear space it suffices to define convergence to zero.

Definition. *Weak convergence*: Let $\{T_j\}$ be a sequence in $\mathscr{D}'$. We say that T_j converges to zero in the sense of distributions, or $T_j \to 0$ in $\mathscr{D}'$, if $\langle T_j, \varphi\rangle \to 0$ for every $\varphi \in \mathscr{D}$.

Strong convergence requires a certain uniformity and it will be defined when needed.

Theorem. Differentiation is a continuous operation, i.e. $T_j \to 0$ in $\mathscr{D}'$ implies that $T_j' \to 0$ in $\mathscr{D}'$.

Proof. $\langle T'_j, \varphi\rangle = -\langle T_j, \varphi'\rangle \to 0$ for every $\varphi \in \mathscr{D}$.

Remarks. The weak topology defined here makes convergent a lot of sequences which are ordinarily divergent. A series which is convergent in the sense of distributions may be differentiated term by term, i.e., if $T = \sum T_j$ then $T' = \sum T'_j$.

Theorem. Let $f_j \to 0$ almost everywhere and suppose that $|f_j| \leqslant g$, where g is a fixed positive locally integrable function. Then $f_j \to 0$ in the sense of distributions.

Proof. This follows from the Lebesgue convergence theorem.

Example. A trigonometric series

$$\sum_k a_k\, e^{2\pi ikx}$$

is convergent in the sense of distributions if and only if $|a_k| \leqslant Ak^\alpha$, for $k \neq 0$, where A is a constant and α is some positive integer. Thus many trigonometric series become convergent in the sense of distributions. To see this consider the series

$$\sum_{k\neq 0} \frac{a_k}{(2\pi ik)^{\alpha+2}}\, e^{2\pi ikx}.$$

It is uniformly convergent since

$$\frac{|a_k|}{|2\pi ik|^{\alpha+2}} \leqslant \frac{A}{(2\pi)^{\alpha+2}}\frac{1}{k^2}.$$

Therefore this series converges also in the sense of distributions. If we differentiate now $\alpha + 2$ times term by term we obtain the original series which therefore is convergent in the sense of distributions.

Examples. The series

$$\sum_{k=-\infty}^{\infty} e^{2\pi ikx}$$

is ordinarily divergent. However, in the sense of distributions it converges to the distribution

$$\sum_{k=-\infty}^{\infty} \delta(x-k)$$

δ	δ	δ	δ	δ
-2	-1	0	1	2

Differentiating term by term we see that

$$\sum_{-\infty}^{\infty} (2\pi ik)\, e^{2\pi ikx}$$

converges to

$$\sum_{k=-\infty}^{\infty} \delta'(x-k)$$

δ'	δ'	δ'	δ'	δ'
-2	-1	0	1	2

Affine Spaces: Lorentz Transformations

In the previous section we defined the space $\mathscr{D}'(R^n)$ of distributions on the Euclidean space R^n. In a similar way we may define the space $\mathscr{D}'(\mathbf{E}_n)$ of distributions on the n-dimensional vector space $\mathbf{E}_n$. However, in physical space there is no pre-determined origin, so that we do not have an $\mathbf{E}_n$ to start with. For this reason, we introduce the concept of an affine space.

Definition. An *affine space* is a set E and an associated vector space $\mathbf{E}$. This association is defined by a map from $E \times E$ to $\mathbf{E}$ which maps a pair a, b of elements of E to the vector $\mathbf{ab}$ of E, and such that the following two laws are satisfied:

(1) *Chasles' relation*: If a, b, c are any three elements of E, then $\mathbf{ab} + \mathbf{bc} + \mathbf{ca} = 0$.

(2) Let o be a fixed element of E. The map $a \rightarrow \mathbf{oa}$ is a one-to-one correspondence between E and $\mathbf{E}$.

It should be noted that (1) may be generalized to more than three elements. Furthermore, according to (1) the triple a,a,a yields $3\,\mathbf{aa} = 0$ or $\mathbf{aa} = 0$ and the triple a,a,b yields $\mathbf{ab} + \mathbf{ba} = 0$.

For obvious reasons, the notation

$$\mathbf{ab} = \overrightarrow{b - a}$$

is very convenient. Thus the difference between two elements a,b of E is a map which maps the pair a,b to the vector $\mathbf{ab}$ of $\mathbf{E}$, and which obviously satisfies the above two laws. If a is a given element of E and $\mathbf{x}$ is a given element of $\mathbf{E}$, then there exists one and only one element $b \in E$ such that $a + \mathbf{x} = b$ where this equality is equivalent to $\mathbf{x} = \overrightarrow{b - a}$.

Definition. Let E and F be two affine spaces. The map

$$\sigma : E \to F$$

is called an *affine operator* from E to F if there exists an associated linear operator

$$\boldsymbol{\sigma} : \mathbf{E} \to \mathbf{F}$$

such that

$$\overrightarrow{\sigma d - \sigma a} = \boldsymbol{\sigma}(\overrightarrow{b - a}).$$

Note that the associated linear operator $\boldsymbol{\sigma}$ is uniquely determined by σ. Furthermore, the composition of two affine operators is an affine operator and the invertible affine operators form a group.

Example. The translation $\mathbf{U} : x \to x + \mathbf{U}$ is an affine operator from the affine space E onto itself. The associated linear operator of a translation is the identity operator,

$$\overrightarrow{(b + \mathbf{U}) - (a + \mathbf{U})} = \overrightarrow{b - a}$$

Conversely, every affine operator having the identity as its associated linear operator is a translation.

Let $\mathbf{E}$ be a vector space over the reals and consider a quadratic form $(\mathbf{x}|\mathbf{y})$ defined on $\mathbf{E}$. It is assumed that $(\mathbf{x}|\mathbf{y})$ is bilinear, symmetric $((\mathbf{x}|\mathbf{y}) = (\mathbf{y}|\mathbf{x}))$ and non-degenerate (no element except zero is orthogonal to the whole space).

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be an orthonormal basis in $\mathbf{E}$, i.e., $(\mathbf{e}_i|\mathbf{e}_j) = 0$ for $i \neq j$ and $(e_i|e_i) = \pm 1$. Every finite dimensional vector space with a non-degenerate quadratic form has an infinite number of orthonormal bases. However, the number of basis elements e such that $(e|e) = +1$

and the number of basis elements e such that $(e|e) = -1$ is independent of the particular chosen basis.

Definition. The *signature* of an n-dimensional vector space with respect to a given quadratic form $(x|y)$ is the pair of integers (p, q), where $p + q = n$, p is the number of O.N. basis elements e such that $(e|e) = +1$ and q is the number of O.N. basis elements e such that $(e|e) = -1$.

Definition. A *Lorentz four-dimensional vector space* is a vector space with a quadratic form which has the signature (3,1). The orthonormal basis will be denoted by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_0$, where $(e_i|e_i) = +1$, $i = 1,2,3$ and $(\mathbf{e}_0|\mathbf{e}_0) = -1$. A *Lorentz four-dimensional affine space* is an affine space E_4 whose associated vector space $\mathbf{E}_4$ has the signature (3,1). By a *Galilean reference system* we mean a chosen origin 0 in E_4 and a chosen orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_0$ in $\mathbf{E}_4$ (chosen coordinate system).

Every point of the universe has four coordinates x_1, x_2, x_3, $x_0 = ct$, three of space and one of time.

Definition. A *Lorentz transformation* σ is an affine invertible operator in a Lorentz affine space which preserves its Lorentz structure, i.e., the associated linear operator preserves the quadratic form

$$(\boldsymbol{\sigma}\mathbf{x}|\boldsymbol{\sigma}\mathbf{y}) = (\mathbf{x}|\mathbf{y}).$$

The Lorentz transformations form a group. The group G consisting of all the Lorentz transformations σ will be called the *inhomogeneous Lorentz group*, whereas the group $\mathbf{G}$ consisting of the associated linear operators $\boldsymbol{\sigma}$ will be called the *homogeneous Lorentz group*.

Example. Translations are Lorentz transformations.

One may now define the space $\mathscr{D}'(E)$ of distributions over the affine space E. More generally, one can define the space $\mathscr{D}'$ for a manifold V, as the space of infinitely differentiable functions with compact support, with a suitable topology, and $\mathscr{D}'(V)$, space of distributions on V, as its dual.

Universal Scalar Particles

Now the definitions of a scalar particle and a universal particle will be made more precise.

Definition. A *scalar particle* in the universe E_4 is a set $\mathscr{H}$ satisfying the postulates:

(1) $\mathscr{H}$ is a vector subspace of $\mathscr{D}'(E_4)$.

(2) $\mathscr{H}$ is equipped with a Hilbertian structure, that is, there is a linear-antilinear form $(\psi_1|\psi_2)_{\mathscr{H}}$ (linear in ψ_1 and antilinear in ψ_2) in $\mathscr{H}$ which is Hermitian and positive definite, and $\mathscr{H}$ is complete with respect to the norm $\|\psi\|_{\mathscr{H}} = (\psi|\psi)^{\frac{1}{2}}_{\mathscr{H}}$.

(3) The canonical embedding of $\mathscr{H}$ into $\mathscr{D}'$ is continuous, that is,

$$\psi_j \to 0 \text{ in } \mathscr{H} \Rightarrow \psi_j \to 0 \text{ in } \mathscr{D}'.$$

We shall find that $\mathscr{H}$ represents charged particles. If the distributions in E_4 were restricted to be real valued, then $\mathscr{H}$ would describe a neutral particle.

Definition. A motion of a particle is an element $\psi \in \mathscr{H}$ such that $\|\psi\|_{\mathscr{H}} = 1$.

A universal particle (universal with respect to the Lorentz group) is one which is considered the same by different observers. An observer makes his observations in some frame of reference; thus the particle $\mathscr{H}$ is interpreted by him as being a space of distributions over R^4 instead of E_4. If all observers interpret $\mathscr{H}$ to be the same space of distributions over R^4, then $\mathscr{H}$ is a universal particle. A more precise definition is given after the operation of $\sigma \in G$ on distributions is defined.

A Lorentz transformation $\sigma \in G$ not only operates on E_4 but also on every structure given over E_4. If $\varphi(x)$ for $x \in E_4$ is a complex function on E_4, the transformation $\varphi \to \sigma\varphi$ is defined by the equation

$$\sigma\varphi(\sigma x) = \varphi(x) \qquad x \in E_4$$

or, equivalently,

$$\sigma\varphi(y) = \varphi(\sigma^{-1}y) \qquad y \in E_4.$$

From the fact that $\boldsymbol{\sigma} \in \mathbf{G}$ is a linear operator, it follows that:

Theorem. $\varphi \in \mathscr{D}(E_4) \Rightarrow \sigma\varphi \in \mathscr{D}(E_4)$. It follows from the definition of $\sigma\varphi$ that:

Theorem. $\varphi_n \to 0 \Rightarrow \sigma\varphi_n \to 0$.

Thus σ gives an automorphism of $\mathscr{D}$ onto $\mathscr{D}$.

The operation of σ on distributions is defined by the equation

$$\langle \sigma T, \sigma\psi \rangle = \langle T, \varphi \rangle$$

or, equivalently,

$$\langle \sigma T, \psi \rangle = \langle T, \sigma^{-1}\psi \rangle = \langle T_y, \psi(\sigma y).$$

Theorem. The operator σ operates linearly and continuously on distributions.

Proof. Linearity:

$$\langle \sigma(a_1 T_1 + a_2 T_2), \psi \rangle = \langle a_1 T_1 + a_2 T_2, \sigma^{-1}\psi \rangle$$
$$= a_1 \langle T_1, \sigma^{-1}\psi \rangle + a_2 \langle T_2, \sigma^{-1}\psi \rangle$$
$$= a_1 \langle \sigma T_1, \psi \rangle + a_2 \langle \sigma T_2, \psi \rangle = \langle a_1 \sigma T_1 + a_2 \sigma T_2, \psi \rangle.$$

Continuity: Given $T_n \to 0$, then for any $\psi \in \mathscr{D}$, we have $\sigma^{-1}\psi \in \mathscr{D}$ and

$$\langle \sigma T_n, \psi \rangle = \langle T_n, \sigma^{-1}\psi \rangle \to 0$$

therefore $\sigma T_n \to 0$.

It is simple to show that the operation of σ followed by τ on $\mathscr{D}'$ is the same as the operation of $\tau\sigma$ on $\mathscr{D}'$. Then it follows that σ is an automorphism of $\mathscr{D}'$ onto $\mathscr{D}'$.

Given an affine space E and a positive measure on **E** which is invariant under translation, a measure on E is uniquely defined.

Then any locally integrable function f on E defines a distribution

$$\langle f, \varphi \rangle = \int f(x)\, \varphi(x)\, dx.$$

Given a quadratic form on **E**, there corresponds orthonormal bases and a Haar measure. In view of the fact that any $\boldsymbol{\sigma} \in \mathbf{G}$ preserves the quadratic form, it will also preserve the Haar measures. It follows that σ preserves the correspondence between functions and distributions for this measure.

Given a scalar particle $\mathscr{H} \subset \mathscr{D}'(E_4)$ and any $\sigma \in G$, one may form the space $\sigma\mathscr{H}$, the set of $\sigma\psi$ for all $\psi \in \mathscr{H}$. With the inner product

$$(\sigma\psi_1 | \sigma\psi_2)_{\sigma\mathscr{H}} = (\psi_1 | \psi_2)_{\mathscr{H}}$$

the space $\sigma\mathscr{H}$ is also a Hilbert space.

Definition. A scalar particle $\mathscr{H}$ is *universal* if for all $\sigma \in G$ the following is true:

(1) $\sigma\mathscr{H} = \mathscr{H}$

(2) $\|\sigma\psi\|_{\mathscr{H}} = \|\psi\|_{\mathscr{H}}$ for all $\psi \in \mathscr{H}$.

It follows that $\mathscr{H}$ is a universal particle if and only if every $\sigma \in G$ is a unitary operator of $\mathscr{H}$ onto $\mathscr{H}$.

Scalar and Vector Particles in an Arbitrary Universe

Definition. A *universe* V is a C^∞-manifold of finite dimension n. A group G whose elements operate on V will be called the structure group of the universe.

Definition. A *scalar particle in the universe* V is a set $\mathscr{H}$ satisfying the postulates:

(1) $\mathscr{H}$ is a vector subspace of $\mathscr{D}'(V)$, the space of distributions in V.

(2) $\mathscr{H}$ is equipped with a Hilbertian structure.

(3) $\psi_j \to 0$ in $\mathscr{H} \Rightarrow \psi_j \to 0$ in $\mathscr{D}'(V)$.

Definition. A scalar particle $\mathscr{H}$ in the universe V is *universal* (with respect to G) if for all $\sigma \in G$:

(1) $\sigma \mathscr{H} = \mathscr{H}$

(2) $\|\sigma\psi\|_{\mathscr{H}} = \|\psi\|_{\mathscr{H}}$ for all $\psi \in \mathscr{H}$.

Example. For one scalar particle, we may take $V = E_4$ with a given Lorentz quadratic form and the corresponding Lorentz group as the structure group.

Example. For two particles, we take $V = E_4 \times E_4$. The structure group G is again the Lorentz group acting on $E_4 \times E_4$ as follows: For $(x, y) \in E_4 \times E_4$ and $\sigma \in G$

$$(x, y) \to \sigma(x, y) = (\sigma x, \sigma y).$$

In order to treat particles such as the electron, proton, etc., we must introduce the concept of a vector-valued distribution. Let $\mathbf{F}$ be a finite dimensional vector space over C.

Definition. *An* $\mathbf{F}$*-valued distribution* $\mathbf{T}$ *on* V is a continuous linear map $\mathbf{T}: \varphi \to \langle \mathbf{T}, \varphi \rangle$ of $\mathscr{D}(V)$ into $\mathbf{F}$.

The space $\mathscr{D}'(V; \mathbf{F})$ of $\mathbf{F}$-valued distributions on V, the space $\mathscr{L}(\mathscr{D}(V); \mathbf{F})$ of continuous linear maps of $\mathscr{D}(V)$ into $\mathbf{F}$, and the tensor product $\mathscr{D}'(V) \otimes \mathbf{F}$ of $\mathscr{D}'(V)$ and $\mathbf{F}$ are all identical:

$$\mathscr{D}'(V; \mathbf{F}) = \mathscr{L}(\mathscr{D}(V); \mathbf{F}) = \mathscr{D}'(V) \otimes \mathbf{F}.$$

Example. Let $V = R^n$ be an affine space with a Lebesgue measure. If $\mathbf{f}(x)$ is a locally integrable $\mathbf{F}$-valued function on R^n, then to $\mathbf{f}$ corresponds a distribution

$$\varphi \to \langle \mathbf{f}, \varphi \rangle = \int \mathbf{f}(x)\, \varphi(x)\, dx.$$

If $S \in \mathscr{D}'(V)$ and $\mathbf{f} \in \mathbf{F}$, then the vector-valued distribution $S\mathbf{f} \in \mathscr{D}'(V; \mathbf{F})$ may be defined by the equation

$$\langle S\mathbf{f}, \varphi \rangle = \langle S, \varphi \rangle \mathbf{f}.$$

$S\mathbf{f}$ is identified with $S \otimes \mathbf{f} \in \mathscr{D}'(V) \otimes F$.

If $\mathbf{F}$ has the basis

$$\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$$

then $\mathbf{T} \in \mathscr{D}'(V; \mathbf{F})$ can be written

$$\mathbf{T} = T_1\mathbf{f}_1 + T_2\mathbf{f}_2 + \ldots + T_n\mathbf{f}_n$$

where $T_1, T_2, \ldots, T_n \in \mathscr{D}'(V)$. Thus for $\varphi \in \mathscr{D}(V)$:

$$\langle \mathbf{T}, \varphi \rangle = \sum_{i=1}^{n} \langle T_1, \varphi \rangle \mathbf{f}_i.$$

Definition. An **F**-*valued particle in the universe* V is a set $\mathscr{H} \subset \mathscr{D}'(V; \mathbf{F})$ satisfying the same postulates as a scalar particle in the universe V except that $\mathscr{D}'(V)$ is replaced by $\mathscr{D}'(V; \mathbf{F})$ in the definition and the following additional postulates are satisfied:

(1) Every $\sigma \in G$ operates not only on V, but also on $\mathbf{F}$, thus $x \in V \Rightarrow \sigma x \in V$, and $\mathbf{f} \in \mathbf{F} \Rightarrow \sigma\mathbf{f} \in \mathbf{F}$.

(2) If σ defines the identity operation in both V and $\mathbf{F}$, then σ is the identity of G.

Remark. G operates faithfully on the product $V \otimes \mathbf{F}$, but not necessarily on V or $\mathbf{F}$ alone.

Example. For an electron, G is the proper spinor group, $V = E_4$, and $\mathbf{F}$ is a two-dimensional vector space over C. There is a mapping $\sigma \in G \rightarrow \sigma_0 \in$ proper inhomogeneous Lorentz group such that two elements of G correspond to each element in the Lorentz group, and the action of each σ on any element of E_4 is the same as the action of the corresponding σ_0 given by the mapping. There is also a mapping $\sigma \in G \rightarrow \tau \in$ the set of unimodular operators in $\mathbf{F}$, which form a group, such that an infinite number of elements of G correspond to each element of this group of linear operators in $\mathbf{F}$, and the action of each σ on any element of $\mathbf{F}$ is the same as the action of the corresponding τ given by the mapping.

Definition. The operation of σ on $\mathbf{T} \in \mathscr{D}'(V) \otimes F$ is defined by the equation

$$\sigma(\langle \mathbf{T}, \varphi \rangle) = \langle \sigma\mathbf{T}, \sigma\varphi \rangle,$$

or, equivalently,

$$\langle \sigma\mathbf{T}, \psi \rangle = \sigma(\langle \mathbf{T}, \sigma^{-1}\psi \rangle) = \tau(\langle \mathbf{T}, \psi(\sigma_0 x) \rangle).$$

Definition. A *universal* **F**-valued particle in the universe V is defined in exactly the same way as a *universal* scalar particle in the universe V.